# Equilibrium-Independent Passivity: a New Definition and Implications

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***Abstract*— We extend the traditional notion of passivity to a forced system whose equilibrium is nontrivially dependent on the control input by defining *equilibrium-independent passivity*, a system property characterized by a dissipation inequality centered at an arbitrary equilibrium point. We also provide input/output tests to certify this property which can be performed analytically or empirically. An example from network stability analysis is presented which demonstrates the utility of this new definition. Finally, through a numerical example we show that equilibrium-independent passivity is less restrictive than incremental passivity.**

## I. Introduction and Motivation

Since Willems' seminal paper on the topic in 1972 [1], dissipation inequalities have been studied extensively as a means for reducing high-order systems to interconnections of manageable subsystems in order to more easily ascertain the behavior of the full system. Specifically, the particular dissipation inequality associated with passivity has proven useful in analyzing cascade and feedback systems, even in the nonlinear case when the system dynamics are suitably structured [2], [3].

We define a new system property, called *equilibrium-independent passivity*, characterized by a dissipation inequality which is referenced to an arbitrary equilibrium input/output pair by construction of the storage function. We will show in Section III that this new property is preserved under various common and useful system interconnections in a manner analogous to standard passivity. In Section IV we present our main result: a set of sufficient conditions on system dynamics and, more importantly, input/output behavior which guarantee that a system possesses this property. In Section V we use these sufficient conditions and the interconnection properties of Section III to simplify the analysis of a network flow.

This property is particularly useful in the analysis of interconnected systems, where the network equilibrium depends on all components and is highly uncertain. Indeed, it was used implicitly in recent studies of reaction networks: the ideas that are defined, framed, and formalized in this paper were presented and justified as *ad hoc* assumptions in [4]–[6]. Our goal here is to examine these assumptions for their own sake and group them as a new system property.

Equilibrium-independent passivity involves a dissipation inequality referenced to two system trajectories, one of which is a fixed point. As such, it may be referred to as an incremental property, but the reader should be careful to observe (and this issue will be discussed in Section VI) that the properties presented here are *not* equivalent to the property of *incremental passivity* as traditionally defined ( [7], [8]), with equilibrium-independent passivity being less restrictive. The work of Jayawardhana and his colleagues in [9], [10] examines the relationship between passivity and incremental passivity in detail and derives conditions under which a passive system is also incrementally passive. We will instead simply show that our notion of equilibrium-independent passivity does not by itself imply incremental passivity.

Since the ideas that motivated this study have proven successful in analyzing a certain class of feedback interconnections that occur regularly in biological systems and communication networks, it is our hope that by elevating this dissipation relationship to the level of a system property we will encourage the search for other classes of systems which may be effectively analyzed with it.

## II. Notation and Assumptions

The 2-norm will be used exclusively. Throughout we will use $\Sigma$ to refer to a general dynamical system of the form

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

with

$$x(t) \in \mathcal{X} \subseteq \mathbb{R}^n, \ u(t) \in \mathcal{U} \subseteq \mathbb{R}^m, \ y(t) \in \mathcal{Y} \subseteq \mathbb{R}^m.$$

When special structure (e.g. SISO or scalar state) is imposed on $\Sigma$ it will be made clear.

We assume that there exists a set $\mathcal{U}^* \subseteq \mathcal{U}$ such that for every $u^* \in \mathcal{U}^*$ there exists a unique $x^* \in \mathcal{X}$ such that $f(x^*, u^*) = 0$. We then define

$$k_x(u) : \mathcal{U}^* \to \mathcal{X} \text{ such that } x^* = k_x(u^*) \qquad (1)$$

and assume it to be at least once differentiable. We call this the *equilibrium input/state map*.

Supported in part by the National Science Foundation under grant ECCS 0852750, the Air Force Office of Scientific Research under grant FA9550-09-1-0092, and NASA Langley NRA contract NNH077ZEA001N entitled "Analytical Validation Tools for Safety Critical Systems." The technical contract monitor is Dr. Christine Belcastro.

We also define the once-differentiable *equilibrium input/output map*

$$k_y(u) : \mathcal{U}^* \to \mathcal{Y} \tag{2}$$

by

$$y^* = k_y(u^*) = h(k_x(u^*), u^*) \tag{3}$$

These assumptions are central to the ideas of this paper, and will together be referred to as the *basic assumption*. The assumption of the existence of the equilibrium input/output map is not restrictive in practice, especially when the output has no feedthrough: after assuming invertibility of $k_x$, the invertibility of $k_y$ is in this case equivalent to monotonicity of $h$.

Finally we state for completeness the working definition of monotonicity and co-coercivity. A function $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is *monotonically increasing* on $\mathcal{D}_0 \subseteq \mathcal{D}$ if

$$\left(f(x) - f(y)\right)^T (x - y) \geq 0 \tag{4}$$

for all $x, y \in \mathcal{D}_0$. If the inequality is reversed $f$ is monotonically decreasing. Given a constant $\gamma > 0$, the function $f(\cdot)$ is called $\gamma$-*co-coercive* on $\mathcal{D}_0$ if

$$\left(f(x) - f(y)\right)^T (x - y) \geq \gamma \|f(x) - f(y)\|^2 \tag{5}$$

for all $x, y \in \mathcal{D}_0$. The definition of monotonicity presented here is adapted from [11], and the definition of $\gamma$-co-coercivity is adapted from [12].

## III. A New Definition

We are now ready to state the definitions that are to be examined in detail.

*Definition 1:* $\Sigma$ is *equilibrium-independent passive* (EIP) on $\mathcal{U}^*$ if for every $u^* \in \mathcal{U}^*$ there exists a once-differentiable and positive definite storage function $S_{u^*} : \mathcal{X} \to \mathbb{R}$ such that $S_{u^*}(x^*) = 0$ and

$$\nabla_x S_{u^*} \cdot f(x, u) \leq (u - u^*)^T (y - y^*) \tag{6}$$

for all $u \in \mathcal{U}$, $x \in \mathcal{X}$. It is understood that $y = h(x, u)$ and $y^* = k_y(u^*)$. We will henceforth make the notational convenience of dropping the subscript from the storage function and referring to it simply as $S$. For brevity we will also use the shorthand $\dot{S}$ to denote $\nabla S \cdot f(x, u)$.

*Definition 2:* $\Sigma$ is *output strictly* EIP (OSEIP) on $\mathcal{U}^*$ if

$$\dot{S} \leq (u - u^*)^T (y - y^*) - \rho(y - y^*) \tag{7}$$

for some positive definite function $\rho(\cdot)$.

The acronym EIP will be used to mean "equilibrium-independent passive" or "equilibrium-independent passivity" depending on the grammatical context.

We first prove the following useful fact:

*Lemma 1:* If $\Sigma$ is EIP, then $k_y(u)$ is monotonically increasing. If $\Sigma$ is OSEIP with $\rho = \gamma\|y - y^*\|^2$, then $k_y(u)$ is $\gamma$-co-coercive.

*Proof:* Select an arbitrary constant input $u^*$, leading to an equlibrium pair $(x^*, u^*)$, where $x^* = k_x(u^*)$. Then there is an $S(x)$ with $S(x^*) = 0$ and $S(x) > 0$ for $x \neq x^*$ such that

$$\dot{S}(x, u) \leq (u - u^*)^T (y - k_y(u^*)) \ \forall \ x, u. \tag{8}$$

In particular, if we select an arbitrary constant $\tilde{u}$, this relationship holds for $(k_x(\tilde{u}), \tilde{u})$, at which $\dot{S} = 0$:

$$\begin{aligned} \dot{S}(k_x(\tilde{u}), \tilde{u}) &= \nabla S \cdot f(k_x(\tilde{u}), \tilde{u}) \\ &= 0 \leq (\tilde{u} - u^*)^T (k_y(\tilde{u}) - k_y(u^*)) \end{aligned} \tag{9}$$

This inequality is the functional definition of monotonicity of $k_y$. $\diamond$

Using the same approach with the supply rate

$$(u - u^*)^T (y - y^*) - \gamma\|y - y^*\|^2$$

gives the inequality

$$0 \leq (\tilde{u} - u^*)^T (k_y(\tilde{u}) - k_y(u^*)) - \gamma\|k_y(\tilde{u}) - k_y(u^*)\|^2 \tag{10}$$

or

$$(\tilde{u} - u^*)^T (k_y(\tilde{u}) - k_y(u^*)) \geq \gamma\|k_y(\tilde{u}) - k_y(u^*)\|^2 \tag{11}$$

which is the definition of $\gamma$-co-coercivity of $k_y$. ■

*Remark 1:* This lemma demonstrates that $\gamma$-co-coercivity of $k_y(u)$ is a necessary condition for OSEIP of $\Sigma$ with $\rho = \gamma\|y - y^*\|^2$; with $\gamma = 0$ we recover monotonicity, which was shown to be necessary for EIP. We call the parameter $\gamma$ the "excess parameter", as it quantifies the output strictness of a dynamical system's EIP, or the excess of EIP.

We now give some basic results for EIP systems.

*Example 1:* A linear (output strictly) passive system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with $A$ invertible, $(A, B)$ controllable, and $(A, C)$ observable is (OS)EIP. This may be shown by direct application of the KYP lemma as in §6.4 of [13] using

$$S(x) = \frac{1}{2}(x - x^*)^T P (x - x^*)$$

as the storage function. The derivation is routine and is not central to the discussion at hand, but it is germane to point out that invertibility of $A$ is required to satisfy the basic assumption.

*Example 2:* The parallel interconnection of two EIP systems $\Sigma_1$ and $\Sigma_2$ as shown in Figure 1 is EIP. Label as $y_i$ the output of $\Sigma_i$. Then since $\Sigma_1$ and $\Sigma_2$ are EIP,

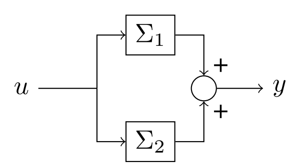

Fig. 1. Parallel interconnection of two EIP systems.

$$\begin{aligned} \dot{S}_1 &\leq (y_1 - y_1^*)^T (u - u^*) \\ \dot{S}_2 &\leq (y_2 - y_2^*)^T (u - u^*) \end{aligned}$$

Observing that $y_1 = y - y_2$, we have

$$\frac{d}{dt}(S_1 + S_2) \leq (y - y^*)^T (u - u^*) \tag{12}$$

so $S_1 + S_2$ is a storage function for the interconnection and the interconnection is EIP. Since $\Sigma_i,\ i = 1, 2$ satisfy the basic assumption, then so does this interconnection.

*Example 3:* The negative feedback interconnection of two EIP systems $\Sigma_1$ and $\Sigma_2$ as shown in Figure 2 is EIP if the interconnection satisfies the basic assumption. The storage function $S_1 + S_2$ again verifies EIP of the interconnection, but in this case the basic assumption must be satisfied in order to write down the equilibrium pair $(u^*, y^*)$ for the interconnection.

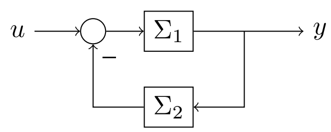


Fig. 2. Negative feedback interconnection of two EIP systems.

*Remark 2:* We define a static nonlinearity to be EIP if

$$0 \le (u - u^*)^T (y - y^*) \tag{13}$$

and OSEIP if

$$0 \le (u - u^*)^T (y - y^*) - \gamma \|y - y^*\|^2 \tag{14}$$

since the storage function of a static block is zero by definition. If any of the systems, say $\Sigma_i$, in an interconnection are static nonlinearities (as opposed to dynamical systems) the interconnection may still be analyzed in this framework by setting the storage function $S_i(x) \equiv 0$ when constructing the storage function of the interconnection as in (13,14). Note that the dissipation inequality describing the static nonlinearity must still be included when constructing the dissipation inequality describing the interconnection.

## IV. Sufficient Conditions for Scalar Systems

Up to this point all results are applicable to a general nonlinear system $\Sigma$. For the remainder of the paper we will consider $\Sigma$ to be a scalar system that is affine in control with no feedthrough:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

The sets in which the input, state, and output evolve are as in Section II with the understanding that they are all contained in $\mathbb{R}$.

With this restriction we can immediately return to Example 3 and show that in the case of scalar state and $\mathcal{U}^* = \mathcal{Y}^* = \mathbb{R}$ the basic assumption holds. To see this, label as $k_y^i$ the steady-state input-output map corresponding to $\Sigma_i$. Then the function $F(y, u^*) = y - k_y^1(u^* - k_y^2(y))$ is uniformly monotone since $F'(y) = 1 + k_y^{1'} \cdot k_y^{2'} \ge 1$, which follows from the EIP of $\Sigma_1$ and $\Sigma_2$. Therefore $F$ is homeomorphic from $\mathbb{R}$ to $\mathbb{R}$ (§5.4.5 of [11]) and a unique solution exists to $F(y, u^*) = 0$, so the steady-state input/ouput map $k_y^{int}(u)$ of the interconnection is invertible and is once differentiable by the implicit function theorem (§5.2.4 of [11]).

We are now ready to state the main result: sufficient conditions relating the dynamics and input/output behavior of a system to its EIP. Along the way we will use the following fact from calculus:

$$k_y'(u) = \left(\frac{dk_x}{du}\right) h'(x) = \left. \frac{h'(x)}{\left(-\frac{f(x)}{g(x)}\right)'} \right|_{x = k_x(u)} \tag{15}$$

*Theorem 1:* Let $\Sigma$ be a scalar affine-in-control system with no feedthrough and a continuous function $g(x) \neq 0$ anywhere, and let $\mathcal{U}^*$ be such that $k_x(\mathcal{U}^*) = \mathcal{X}$. $\Sigma$ is EIP on $\mathcal{U}^*$ if both the following conditions are satisfied:

1) $\text{sign}(g(x)) \cdot h(x)$ is monotonically increasing for all $x \in \mathcal{X}$
2) $0 \le k_y'(u)$ for all $u \in \mathcal{U}^*$.

If additionally

2′) $0 \le k_y'(u) \le \frac{1}{\gamma}$ for all $u \in \mathcal{U}^*$,

then $\Sigma$ is OSEIP with $\rho(\xi) = \gamma \xi^2$.

*Proof:* Condition 1 guarantees that the function

$$S(x) = \int_{x^*}^{x} \frac{h(\sigma) - h(x^*)}{g(\sigma)} \, d\sigma \tag{16}$$

is positive when $x \neq x^*$ and satisfies $S(x^*) = 0$, so $S$ is a storage function candidate. Differentiate to find

$$\dot{S} = \nabla S[f(x) + g(x)u] \tag{17}$$

$$= \frac{h(x) - h(x^*)}{g(x)} [f(x) + g(x)u] \tag{18}$$

$$= [h(x) - h(x^*)] \frac{f(x)}{g(x)} + [h(x) - h(x^*)]u \tag{19}$$

$$= [h(x) - h(x^*)] \left[ u - \left( -\frac{f(x)}{g(x)} \right) \right]. \tag{20}$$

Let $F(x) := -\frac{f(x)}{g(x)}$, observe that $u^* = F(x^*)$, and add and subtract $u^*$:

$$\begin{aligned} \dot{S} &= [h(x) - h(x^*)][u - F(x) + F(x^*) - u^*] \\ &= [h(x) - h(x^*)](u - u^*) \\ &\quad + [h(x) - h(x^*)][F(x^*) - F(x)]. \end{aligned} \tag{21}$$

It remains to show that $[h(x) - h(x^*)][F(x^*) - F(x)] \le 0$.

From (15) and condition 2 we know that $h'(x)$ and $F'(x)$ have the same sign. Since $g(x)$ does not change sign, by condition 1 we know that $h'(x)$ does not change sign, and consequently neither does $F'(x)$. It follows that

$$[h(x) - h(x^*)][F(x^*) - F(x)] \le 0,$$

and $\dot{S} \le [h(x) - h(x^*)](u - u^*)$ as desired. ⋄

Let us now consider $2'$. Equation (15) says that in terms of the system dynamics this is equivalent to assuming that

$$\frac{1}{h'(x)} \left( -\frac{f(x)}{g(x)} \right)' \ge \gamma > 0. \tag{22}$$

Taking (16) as our storage function and comparing (21) to our desired supply rate (7) with $\rho(\xi) = \gamma \xi^2$, we would like to demonstrate that

$$[h(x) - h(x^*)] \, [F(x) - F(x^*)] \ge \gamma [h(x) - h(x^*)]^2 \tag{23}$$

or, equivalently,

$$[h(x)-h(x^*)]\{F(x)-F(x^*)-\gamma[h(x)-h(x^*)]\} \geq 0. \quad (24)$$

To show that (24) indeed holds, we use the $F(x)$ notation and note that the upper bound in condition $2'$ is

$$\frac{F'(x)}{h'(x)} \geq \gamma. \quad (25)$$

Because $\text{sign}(g(x)) \cdot h'(x) \geq 0$, (25) means:

$$\text{sign}(g(x)) \cdot F'(x) \geq \gamma \ \text{sign}(g(x)) \cdot h'(x).$$

Consider the quantity

$$\text{sign}(g(x)) \left( \int_{x^*}^{x} F'(\sigma) - \gamma h'(\sigma) \ d\sigma \right) = \text{sign}(g(x)) \left( F(x) - F(x^*) - \gamma[h(x) - h(x^*)] \right). \quad (26)$$

Because of (25), the left-hand side of (26) will always take on the same sign as $x - x^*$, which by assumption of monotonicity of $\text{sign}(g(x)) \cdot h(x)$ will have the same sign as

$$\text{sign}(g(x)) \cdot [h(x) - h(x^*)].$$

Therefore

$$[h(x) - h(x^*)] \{F(x) - F(x^*) - \gamma[h(x) - h(x^*)]\} \geq 0$$

and (24) holds. ■

*Remark 3:* Conditions 2 and $2'$ are also necessary for EIP and OSEIP, respectively, by the scalar case of Lemma 1.

*Remark 4:* The function $k_y(u)$ is the map from constant inputs to steady-state outputs, which can be constructed analytically with knowledge of the system dynamics. However if the system is one which can easily be manipulated and measured in a laboratory setting, then $k_y$ may also be constructed empirically with no knowledge of the dynamics of the system, notably without knowledge of $f(x)$ or detailed knowledge of $h(x)$. The sign of $g(x)$ and the monotonicity of $h(x)$ must of course be verified to apply the main theorem.

## V. EIP-Based Network Stability Analysis

We now consider a network of EIP systems $\Sigma_i$, $i = 1, \cdots, N$, with input $u_i \in R^m$, output $y_i \in R^m$, and storage function $S_i(x_i)$ satisfying (7) with $\gamma_i \geq 0$, and suppose they are coupled according to the feedback law:

$$u = (K \otimes I_m) y, \quad (27)$$

where $K \in R^{N \times N}$, $I_m$ is the $m \times m$ identity matrix, $u := [u_1^T \cdots u_N^T]^T$ and $y := [y_1^T \cdots y_N^T]^T$.

If an equilibrium exists for the network, we check its stability by investigating whether there exists a diagonal matrix $D > 0$ such that

$$D(K - \Gamma) + (K - \Gamma)^T D \leq 0, \quad (28)$$

where

$$\Gamma := \text{diag}(\gamma_1, \cdots, \gamma_N). \quad (29)$$

If such a matrix $D$ exists, its diagonal entries serve as weights in the composite Lyapunov function

$$V = \sum_{i=1}^{N} d_i S_i(x_i), \quad (30)$$

whose time derivative satisfies:

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{N} d_i \dot{S}_i & (31) \\ &\leq \sum_{i=1}^{N} d_i [(u_i - u_i^*)^T (y_i - y_i^*) - \gamma_i \|y_i - y_i^*\|^2] & (32) \\ &= (y - y^*)^T (D \otimes I_m)(u - u^*) \\ &\quad - (y - y^*)^T ((D\Gamma) \otimes I_m)(y - y^*) & (33) \\ &= (y - y^*)^T (D \otimes I_m)(K \otimes I_m)(y - y^*) \\ &\quad - (y - y^*)^T ((D\Gamma) \otimes I_m)(y - y^*) & (34) \\ &= (y - y^*)^T ((D(K - \Gamma)) \otimes I_m)(y - y^*) \leq 0, & (35) \end{aligned}$$

thus proving stability. If some of the blocks are static, the corresponding storage functions $S_i$ in the analysis above must be interpreted as zero as in (13,14).

If (28) holds with strict inequality and if each block has the property that $y_i = y_i^*$ implies $x_i = x_i^*$, then we conclude asymptotic stability. Less restrictive asymptotic stability conditions can be obtained by making use of negative terms of the state that may be present in $\dot{S}_i$ in addition to the input and output terms, or by using the LaSalle-Krasovskii Invariance Principle. Finally, if the Lyapunov function constructed in (31) is radially unbounded, we conclude global asymptotic stability.

*Example 4:* As an illustration of the network stability analysis procedure outlined above, we revisit a passivity-based study of Internet congestion control presented in [6], and demonstrate an EIP property used implicitly in [6]. Congestion control algorithms aim to maximize network throughput while ensuring an equitable allocation of bandwidth to the users. In a decentralized congestion control scheme, each link increases its packet drop or marking probability (interpreted as the "price" of the link) as the transmission rate approaches the capacity of the link. Sources then adjust their sending rates based on the aggregate price feedback they receive in the form of dropped or marked packets.

To see the interconnection structure of sources and links, consider a network where packets from sources $i = 1, \cdots, N$ are routed through links $\ell = 1, \cdots, L$ according to a $L \times N$ routing matrix $R$ in which the $(\ell, i)$ entry is 1 if source $i$ uses link $\ell$ and 0 otherwise. Because the transmission rate $z_\ell$ of link $\ell$ is the sum of the sending rates $x_i$ of sources using that link, the vectors of link rates $z$ and source rates $x$ are related by:

$$z = Rx. \quad (36)$$

Likewise, since the total price feedback $q_i$ received by source $i$ is the sum of the prices $p_\ell$ of the links on its path, the vectors $q$ and $p$ are related by:

$$q = R^T p. \tag{37}$$

The congestion control problem as formulated by Kelly *et al.* [14] and studied by numerous other authors (see the excellent reviews [15], [16]) is to design decentralized algorithms for link prices and user sending rates in such a way that the network equilibrium is stable and solves the optimization problem:

$$\max_{x_i \geq 0} \sum_{i=1}^{N} U_i(x_i) \quad \text{s.t.} \quad z_\ell \leq c_\ell \tag{38}$$

where $U_i(\cdot)$ is a concave utility function for source $i$ and $c_\ell$ is the capacity of link $\ell$. One particular algorithm proposed in [14] is:

$$\dot{x}_i = k_i(x_i)(U_i'(x_i) - q_i) \qquad p_\ell = h_\ell(z_\ell) \tag{39}$$

where $k_i(x_i) > 0$ for all $x_i \geq 0$, $U_i'(\cdot)$ is the derivative of the utility function $U_i(\cdot)$, and $h_\ell(\cdot)$ is a monotone penalty function that grows with a steep slope for $z_\ell \geq c_\ell$. The network equilibrium resulting from this algorithm indeed approximates the solution of the Kuhn-Tucker optimality conditions for (38).

We now apply the EIP-based stability analysis procedure outlined above to prove the stability of this equilibrium without relying on the knowledge of its numerical value. We note from the concavity of $U_i(\cdot)$ that $U_i'(\cdot)$ is a decreasing function, and further assume

$$U_i'(x_i) \to \infty \text{ as } x_i \to 0^+ \tag{40}$$

which guarantees invariance of the nonnegative orthant $R^N_{\geq 0}$. We then rewrite the source algorithm as:

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i \quad y_i = x_i, \tag{41}$$

where $f_i(x_i) = k_i(x_i)U_i'(x_i)$, $g_i(x_i) = k_i(x_i)$, $u_i = -q_i$, and conclude EIP from Theorem 1 because $h(x_i) = x_i$ is increasing, and

$$k_y'(u) = \frac{h'(x)}{\left(-\frac{f(x)}{g(x)}\right)} = \frac{1}{-U_i''(x_i)} \geq 0 \tag{42}$$

by the decreasing property of $U_i'(x_i)$. Likewise, the static blocks $p_\ell = h_\ell(z_\ell)$ are EIP because $h_\ell(\cdot)$ is an increasing function. Denoting by $\Sigma_i$, $i = 1, \cdots, N$ the $x_i$ subsystems and by $\Sigma_{N+\ell}$, $\ell = 1, \cdots, L$, the $p_\ell$ subsystems, we note from (36) and (37) that the feedback matrix is given by:

$$K = \begin{bmatrix} 0 & -R^T \\ R & 0 \end{bmatrix}. \tag{43}$$

Because $K$ is skew-symmetric and $\Gamma \geq 0$ is diagonal, (28) holds with $D = I$, from which stability follows.

To prove asymptotic stability, the first option is to show $\gamma_i > 0$ which, in this example, would be achieved with further assumptions on the utility function that guarantee an upperbound on (42). The other option, pursued here, is to take $\gamma_i = 0$ and investigate state-dependent terms in $\dot{S}_i$ that exist in addition to the input-output product. To this end, we apply the storage function construction in the proof of Theorem 1 and obtain:

$$S_i(x_i) = \int_{x_i^*}^{x_i} \frac{h(\sigma) - h(x_i^*)}{g(\sigma)} d\sigma = \int_{x_i^*}^{x_i} \frac{\sigma - x_i^*}{k_i(\sigma)} d\sigma, \tag{44}$$

which yields:

$$\begin{aligned} \dot{S}_i &= (x_i - x_i^*)(U_i'(x_i) + u_i) && (45)\\ &= (x_i - x_i^*)(U_i'(x_i) - U_i'(x_i^*) + u_i - u_i^*) && (46)\\ &= (x_i - x_i^*)(U_i'(x_i) - U_i'(x_i^*)) \\ &\quad + (y_i - y_i^*)(u_i - u_i^*). && (47) \end{aligned}$$

When $U_i'(\cdot)$ is strictly decreasing, the first term in (45) is negative whenever $x_i \neq x_i^*$ and, thus, the time derivative of the Lyapunov function (31) exhibits a sum of such negative terms for each $i$, thus insuring asymptotic stability. Global asymptotic stability follows when $k_i(x_i)$ satisfies an appropriate growth bound that guarantees unbounded growth of $S_i(x_i)$ in (44) as $x_i \to \infty$.

## VI. Comparison to Incremental Passivity

It is clear from the formulation presented by Stan and Sepulchre in [8] that equilibrium-independent passivity is implied by incremental passivity, which is defined formally in [8] as follows (with some minor notational changes for consistency with the preceding):

*Definition 3:* Let $x_a, x_b$ denote two trajectories of a system $\Sigma$ with corresponding inputs and outputs $u_a, u_b$ and $y_a, y_b$. For convenience define $\Delta x = x_a - x_b$, $\Delta u = u_a - u_b$, $\Delta y = y_a - y_b$. $\Sigma$ is incrementally passive if there is a positive definite storage function $S_\Delta$ with $S_\Delta(0) = 0$ such that

$$S_\Delta(\Delta x(T_f)) - S_\Delta(\Delta x(0)) \leq I_{T_f} = \int_0^{T_f} \Delta u(t)^T \Delta y(t)\ dt \tag{48}$$

for all $T_f > 0$ and any pair of trajectories $x_a, x_b$.

A slightly more general notion of incremental passivity is presented by Desoer and Vidyasagar in an operator theoretic setting in [7]. To recover the condition for EIP, set $u_b(t) = u^*$ constant and use $x_a(0) = x_b(0) = k_x(u^*)$. Now $\Delta x = x_a - k_x(u^*)$, $\Delta u = u_a - u^*$, $\Delta y = y_a - k_y(u^*)$ and (48) becomes the integral form of (6).

We demonstrate with a particular numerical example that the converse is not true: if $\Sigma$ is EIP, it is *not* necessarily incrementally passive.

Consider the following affine, scalar $\Sigma$

$$\begin{aligned} \dot{x} &= f(x) + u \\ y &= x^3 \end{aligned}$$

Where $f(x)$ is continuous and is defined to be $f(x) = -\tanh \frac{x}{2}$ on a bounded interval $\mathcal{I}$ including $[-1, 1]$ and strictly decreasing outside $\mathcal{I}$. It is easy to verify based on the conditions of Theorem 1 that $\Sigma$ is EIP with $\mathcal{U}^* = \mathcal{U} = \mathbb{R}$. To show that the system is not incrementally passive, we need only find one pair of trajectories and some final time $T_f$ for which (48) is violated.

To find such a pair, choose as inputs two pure sine waves

$$u_a = \sin t$$
$$u_b = A\sin(\omega t + \phi)$$

and search for $\min_{A,\omega,\phi}\left(\min_{T_f\in\mathcal{T}} I_{T_f}\right)$ with $(A, \omega, \phi) \in \mathcal{S}$. $I_{T_f}$ is computed with $x_a(0) = x_b(0) = 0$, noting that $x = 0$ is an unforced equilibrium of $\Sigma$.

The MATLAB builtin function `fmincon` in `active-set` mode was used to run the search, and `ode45` with a relative tolerance of $10^{-9}$ did the integrations. This particular example simulates for $T_f \in [0, 3]$ with the solution returned at time increments of $0.00015$ (consequently this is the maximum step size that may be taken by the integration solver). Halving the step size did not change the result presented here. The parameter search was conducted from an initial point $A = 1,\ \omega = 1.5,\ \phi = \pi/2$ through the constraint set

$$-1 \le A \le 1$$
$$0 \le \omega \le \infty$$
$$0 \le \phi \le \pi.$$

The solution discovered is on the boundary of the constraint set. With $A = 1,\ \omega \approx 0.255,\ \phi = 0$ the integral $I_{T_f}$ attains its most negative value of about $-0.0413$ at $T_f = 2.952$. Figure 3 shows $I_{T_f}$ over the time interval $[0, 3.2]$ to make clear the behavior of the integral near the minimum. Because the integral attains a negative value, the system is not incrementally passive.

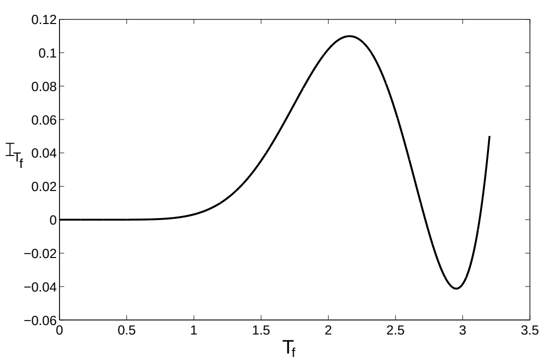


Fig. 3. The integral $I_{T_f}$ of the supply rate as $\Sigma$ evolves over 3.2 units of time.

## VII. Discussion and Future Work

It is our intent to search for broader classes of system for which verification of EIP is reasonably easy and a broader class of interconnections whose behavior can be ascertained more simply if the individual subsystems are EIP.

Immediately we will attempt to formulate sufficient conditions for SISO systems with vector state which gaurantee EIP. The reader will recall that the necessary conditions presented in Lemma 1 are valid for a reasonably general MIMO system, but the sufficient condition is only valid for scalar affine-in-control systems.

There are two immediate extensions for MIMO systems. First is the construction of a storage function (sufficient conditions) when $u$ and $y$ are of the same dimension (greater than 1). The second is a generalization of the supply rate to the form

$$\begin{bmatrix} u - u^* \\ y - y^* \end{bmatrix}^T M \begin{bmatrix} u - u^* \\ y - y^* \end{bmatrix}, \qquad (49)$$

which would include other forms of equilibrium-independent dissipativity as special cases. Notice that this form would also admit unmatched input and output dimensions.

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