Identification of Nonlinear Maps in Interconnected Systems

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***Abstract*— We offer a systematic algorithm for the identification of static nonlinear maps in interconnected systems. The class of systems considered are those consisting of linear time-invariant systems and static nonlinear functions. Under the conditions that the linear dynamics are known and the inputs to the nonlinearities are measurable, the identification problem is reduced to a least squares problem. Notions of identifiability and persistence of excitation are introduced to demonstrate convergence of the estimate to the true nonlinear function under the L_1-norm.**

I. Introduction

The nonlinear system identification problem presents a difficult challenge. Given noisy measurement data, the aim is to reconstruct an estimate of the internal processes that produced the data. In general, these types of inverse problems are ill-posed, and require additional assumptions to be made regarding the solution.

Even with the addition of structural interconnection information, the estimation of arbitrary static nonlinear functions remains an ill-posed problem. In the study of such problems, it is common to assume certain smoothness properties about the solution and employ regularization methods to enforce these properties [1], [2], [3]. This prerequisite of smoothness has also been shown to be an effective tool in the identification of static nonlinear maps in interconnected systems [4], [5]. In this paper we will consider a measure of smoothness that we call the dispersion function, which will impose a regularization based on total variation.

There exists great interest in the systematic inclusion of *a priori* structural information in system identification problems. Information regarding the structural interconnection of the system is generally known with a high degree of confidence due to an understanding of the underlying relationships between the system's components. While the identification of systems with specific interconnections (such as Wiener and Hammerstein systems) has been thoroughly studied [6], [7], [8], [9], [10], a theory for the identification of general interconnected systems has been relatively absent.

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In this paper, we consider those interconnections consisting of finitely many linear time-invariant (LTI) systems and static nonlinear maps. To approach this class of systems, we first make the assumption that the linear dynamics are known.

The methods presented here are nonparametric in the sense that no parameterization is assumed or forced upon the nonlinear functions. The idea of enforcing smoothness in the estimated solution is extended to multi-input nonlinearities and a least squares algorithm is proposed. Furthermore, notions of identifiabilitly and persistence of excitation are formalized and asymptotic convergence of the estimates to the true nonlinear functions is shown.

The remainder of this paper is organized as follows. In Section II, we describe the class of systems under consideration. Section III motivates and describes our metric of staticness, the dispersion function. The identification algorithm is described in Section IV. Section V discusses the notion of identifiability. In Section VI, we explore various properties of input signals that lead to sufficiently rich input-output data sets. Section VII contains our main results regarding convergence.

Notation

x	infinite vector-valued sequence $(x_k)_{k=1}^{\infty}$
$x^{[L]}$	finite vector-valued sequence $(x_k)_{k=1}^{L}$
x_k	k^{th} element of x or $x^{[L]}$
x_k^*	complex-conjugate transpose of x_k
$\mathcal{L}$	known discrete-time LTI system
$\mathcal{N}$	static nonlinear function
$\mathcal{M}(\mathcal{L},\mathcal{N})$	input-output operator of LFT model structure defined by $\mathcal{L}$ and $\mathcal{N}$
$\mathcal{C}$	set of continuous functions on Ω
$\mathbb{L}_\gamma$	class of Lipschitz continuous functions with Lipschitz bound γ.
$\mathbb{N}$	set of static nonlinear functions with a particular input-output structure
$\|D\mathcal{N}\|$	total variation of $\mathcal{N}$
$BV(\Omega)$	functions of bounded variation on Ω
$\mathbb{S}$	$\mathbb{S} = BV(\Omega) \cap \mathbb{N}$
$u_k \in \mathbf{R}^{n_u}$	known input to $\mathcal{L}$
$e_k \in \mathbf{R}^{n_e}$	unknown input to $\mathcal{L}$
$y_k \in \mathbf{R}^{n_y}$	measured output of $\mathcal{L}$
$z_k \in \mathbf{R}^{p}$	input to $\mathcal{N}$
$w_k \in \mathbf{R}^{m}$	output of $\mathcal{N}$
$\hat{w}^{[L]}$	estimate of w
$\mathcal{I}^{[L]}$	interpolant of $(z^{[L]}, w^{[L]})$
$\hat{\mathcal{I}}^{[L]}$	interpolant of $(z^{[L]}, \hat{w}^{[L]})$
$\mathcal{D}^L$	dispersion function

$n^{[L]}$	number of triangles in the L^{th} triangulation
p_i	number of inputs to $\mathcal{N}^{[i]}$
p	$p = \sum_{i=1}^{m} p_i$
$\Omega^{[i]} \subseteq \mathbf{R}^{p_i}$	domain over which $\mathcal{N}^{[i]}$ is to be identified
$\Omega \subseteq \mathbf{R}^{p}$	$\Omega^{[1]} \times \Omega^{[2]} \times \cdots \times \Omega^{[m]}$
$\mathbb{E}$	$\mathbb{E} = \{e : \|e\|_\infty < \infty\}$

$BV(\Omega)$ is the set of functions $\mathcal{N} \in L_1(\Omega)$ with bounded total variation,

$$\|D\mathcal{N}\| = \sup \left\{ \int_\Omega \mathcal{N} \text{ div}\phi \, dx : \phi = (\phi_1, \ldots, \phi_n) \in \mathcal{C}_0^1(\Omega), \right. \\ \left. |\phi(x)| \leqslant 1 \text{ for } x \in \Omega \right\} < \infty,$$

where div $\phi = \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial x_i}$, $\phi : \Omega \to \mathbf{R}^n$, and $\mathcal{C}_0^1(\Omega)$ is the set of $\mathcal{C}^1$ functions on Ω with compact support in Ω. Note that if $\mathcal{N}$ is differentiable, the total variation $\|D\mathcal{N}\|$ becomes $\int_\Omega |\nabla \mathcal{N}| dz$.

II. Problem Formulation

We focus our attention towards those interconnected systems consisting of any finite number of linear time-invariant (LTI) systems and static nonlinear maps. The identification problem we consider is that of identifying the static nonlinear maps in the interconnected system. The linear components of the interconnection are assumed to be known.

To exploit the known interconnection structure, let the system be represented as a Linear Fractional Transformation (LFT) as shown in Figure 1. The LFT is a powerful data

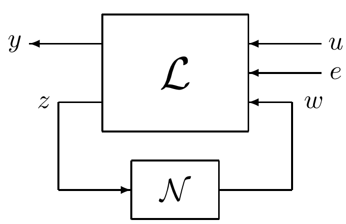

Fig. 1. LFT model structure

structure that can be used to formulate all systems of interest into one common framework [11]. In this paper, we use the LFT to effectively separate the known linear dynamics $\mathcal{L}$ from the unknown static nonlinearities $\mathcal{N}$. We will refer to the generalized system in Figure 1 as the *LFT model structure*.

In general, the nonlinear block $\mathcal{N}$ has a block diagonal structure that we represent as

$$\mathcal{N} = \begin{bmatrix} \mathcal{N}^{[1]} & & \\ & \ddots & \\ & & \mathcal{N}^{[m]} \end{bmatrix}.$$

We can assume without loss of generality that the components of $\mathcal{N}$ are each single-output. This can always be realized by introducing redundant copies of the respective inputs. In some cases, the nonlinear block $\mathcal{N}$ may also contain repeated components. However, we do not consider these cases in this paper. For ease of notation, we will proceed with the analysis with $m = 1$. The generalization to $m > 1$ is straightforward.

The linear block $\mathcal{L}$ may also be partitioned as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{yu} & \mathcal{L}_{ye} & \mathcal{L}_{yw} \\ \mathcal{L}_{zu} & \mathcal{L}_{ze} & \mathcal{L}_{zw} \end{bmatrix}.$$

In addition to our assumption that the linear dynamics are known, we will require that $\mathcal{L}$ possesses the following property.

Definition 1: z is measurable if there exists an LTI system Ψ_M such that

$$\begin{bmatrix} \mathcal{L}_{ze} & \mathcal{L}_{zw} \end{bmatrix} = \Psi_M \begin{bmatrix} \mathcal{L}_{ye} & \mathcal{L}_{yw} \end{bmatrix}.$$

Measurability of z implies that

$$\begin{aligned} z &= \mathcal{L}_{zu} u + \mathcal{L}_{ze} e + \mathcal{L}_{zw} w \\ &= \mathcal{L}_{zu} u + \Psi_M (\mathcal{L}_{ye} e + \mathcal{L}_{yw} w) \\ &= \mathcal{L}_{zu} u + \Psi_M (y - \mathcal{L}_{yu} u). \end{aligned}$$

That is, z can be inferred from u, y and $\mathcal{L}$. Of course, the measurability property can be relaxed if z is already known.

We will equate the selection of a candidate static nonlinear function with specifying a sequence of input-output pairs that represent a sampling of the function. In order for a candidate sequence to possess some validity, we desire that it have the following properties.

1) Consistency with the known data $u, y, \mathcal{L}$.
2) Consistency with *a priori* knowledge regarding the unmeasured signal input e.
3) The sequence must be representative of a sampling from a static function.

The first two properties are standard criterion in system identification problems. The third requirement of staticness is the subject of the following section. Loosely speaking, we relate the staticness of a candidate sequence with the complexity of interpolating the points in the sequence, as measured by a metric we call *the dispersion function*.

III. The Dispersion Function

Consider the following problem. Given input-output data $(u_k, y_k)_{k=1}^{L}$ and a linear system $\mathcal{L}$, we wish to find a static nonlinear function $\mathcal{N}$ that is consistent with the measured data. In general, there are multiple solutions. It is then of interest to develop a metric to aid in the selection of a candidate function among the set of possible solutions. In this section, we develop a particular metric that possesses many computational advantages [4], [5].

Let us examine the scatter plots shown in Figure 2. It is intuitively obvious that the scatter plot on the right is representative of a sampling of a static function whereas the scatter plot on the left is not. We speculate that this intuition is based on a metric of how easily the points can be interpolated using simple functions. One way in which we can realize this metric is to relate the staticness of a scatter plot to the total variation of the graph of its interpolant. We propose to mathematically capture this intuition as follows.

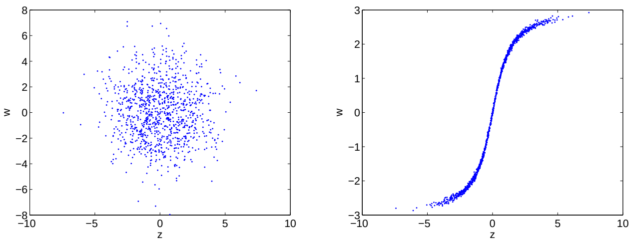

Fig. 2. Data sampled from a dynamic system (left) and static map (right)

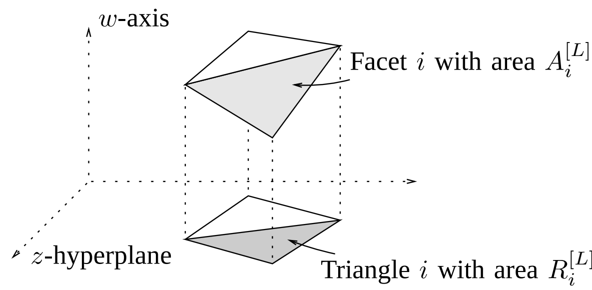

Fig. 4. Triangulation and corresponding facets.

Let us first consider the case where z and w are sequences of scalars. We will assume that no points in z are repeated, i.e., $z_i \neq z_j$ for every $i \neq j$. We define the dispersion function $\mathcal{D}^L$ to be the sum of the squares of the distances between the neighboring points on the scatter plots, scaled by $L-1$. To illustrate the dispersion function, consider the linear interpolant depicted in Figure 3. Here, the (z, w) pairs are sorted so that z is an increasing sequence. The dispersion of this sampling is $4\sum_{i=1}^{4} d_i^2$. For the case where z is vector-

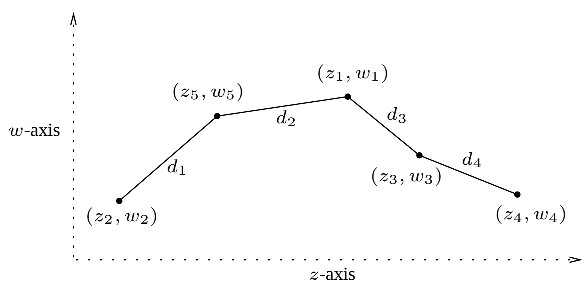

Fig. 3. Illustrating the dispersion function (scalar case).

valued, we can no longer sort the input data as in the scalar case. The analagous operation will be one that associates neighboring points via a triangularization.

Consider a set of points $\{z_k\}_{k=1}^L \subset \mathbf{R}^2$. A triangulation partitions the convex hull of $\{z_k\}_{k=1}^L$ into a disjoint set of triangles whose vertices are points in $\{z_k\}_{k=1}^L$. More generally, for a set of points $\{z_k\}_{k=1}^L \subset \mathbf{R}^p$, a triangulation partitions the convex hull of $\{z_k\}_{k=1}^L$ into a set of simplices. We will also refer to these simplices as "triangles".

Now consider a sequence of data points $(z_k, w_k)_{k=1}^L$ with $z_i \neq z_j$ for every $i \neq j$. If each point $z_k \in \mathbf{R}^p$ is paired with a scalar $w_k \in \mathbf{R}$, corresponding to each triangulation is a faceted interpolant $\mathcal{I}^{[L]}$ (See Figure 4). For the L^{th} triangularization, we denote the area of triangle i to be $R_i^{[L]}$ and the area of the corresponding facet to be $A_i^{[L]}$. While triangulations are not unique, we consider the Delaunay triangulation due to its attractive geometric and computational properties [12]. Our subsequent results, however, do not rely on this choice of triangulation.

When $\{z_k\}_{k=1}^L \subset \mathbf{R}^p, p > 1$, we define the dispersion function as follows.

Definition 2: Consider a set of points $\{z_k\}_{k=1}^L \subset \mathbf{R}^p$ and $\{w_k\}_{k=1}^L \subset \mathbf{R}$. Let $n^{[L]}$ be the number of triangles resulting from a triangularization of $\{z_k\}_{k=1}^L$. The dispersion of $(z^{[L]}, w^{[L]})$ is defined to be

$$\mathcal{D}^L(z^{[L]}, w^{[L]}) = n^{[L]} \sum_{i=1}^{n^{[L]}} \left(A_i^{[L]}\right)^2.$$

It can be shown that the dispersion can be easily computed. Consider the following theorem.

Theorem 1: Let $\bar{w} \in \mathbf{R}^L$ denote a vector whose components are the elements of the sequence $w^{[L]}$. Then, the dispersion function $\mathcal{D}^L$ is quadratic in $\bar{w}$. That is, there exists a matrix $Q \succeq 0$ and a scalar r (both dependent on $z^{[L]}$) such that

$$\mathcal{D}^L(z^{[L]}, w^{[L]}) = \bar{w}^* Q \bar{w} + r.$$

The quadratic nature of the dispersion function will be exploited in the identification algorithm to yield a least squares problem.

There is also an intimate connection between the dispersion of a sampling from a function and the total variation of a function. Of particular importance is the regularization that the dispersion imposes on the iterpolants defined by the sampling.

Lemma 1: If $\mathcal{D}^L(z^{[L]}, w^{[L]}) < M$, then $\mathcal{I}^{[L]}$ is of bounded variation with $\|D\mathcal{I}^{[L]}\| < \sqrt{M}$.

In this paper, we are concerned with the convergence properties of the estimated interpolants $\hat{\mathcal{I}}^{[L]}$. Some of the ideas from parametric identification, namely identifiability and persistence of excitation, have direct analogs. These will be discussed in detail in Sections V and VI. For now, let us consider the following intuitive notions.

1) The LFT model structure is identifiable if it allows for nonlinear functions to be distinguishable through input-output experiments.
2) Let the nonlinearity $\mathcal{N}$ in the LFT model structure be a non-zero function. The input is persistently exciting if the response to this input cannot be zero.

Clearly, if we do not make a restriction on the class of functions that we consider, there is no hope that the above conditions can ever be satisfied and thus no convergence results can be established. It is then necessary to make an assumption about the class of functions that we are considering and ensure that the estimates lie within this class. With this in mind, we consider the class of functions $\mathbb{S} = BV(\Omega) \cap \mathbb{N}$ due to the relationship between the dispersion and total variation of a function.

IV. Identification Algorithm and Analysis Framework

A. The Identification Algorithm

We now propose the following identification algorithm. Define the cost function

$$J^L(w^{[L]}) = \frac{1}{L}\sum_{i=1}^{L}(e_i^{[L]})^* e_i^{[L]} + \beta(L)\mathcal{D}^L(z^{[L]}, w^{[L]}).$$

Here, $\beta(L)$ is a weighting parameter that may be obtained through cross-validation procedures. We can parameterize all possible solutions as

$$\begin{bmatrix} e \\ w \end{bmatrix} = \begin{bmatrix} e^\circ \\ w^\circ \end{bmatrix} + \begin{bmatrix} \mathcal{B}_e \\ \mathcal{B}_w \end{bmatrix} f,$$

where

$$\begin{bmatrix} \mathcal{L}_{ye} & \mathcal{L}_{yw} \end{bmatrix} \begin{bmatrix} \mathcal{B}_e \\ \mathcal{B}_w \end{bmatrix} = 0,$$

(e°, w°) is a particular solution obtained via Kalman Smoothing [4], and f is a free signal. Due to the quadratic nature of the dispersion function, the minimization of J^L can be performed over the free signal f and the identification problem can be formulated as a least squares problem,

$$f^{opt} = \arg\min_f \left\| \begin{bmatrix} \frac{1}{\sqrt{L}} e^\circ \\ \sqrt{\beta(L)} Q^{\frac{1}{2}} w^\circ \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{L}} \mathcal{B}_e \\ \sqrt{\beta(L)} Q^{\frac{1}{2}} \mathcal{B}_w \end{bmatrix} f \right\|^2.$$

B. Analysis Framework

For analysis purposes, we will consider a similar problem with the additional assumption that $\mathcal{L}_{ye}^{-1}$ exists. The optimization problem can then be approached analytically by considering the following formulation.

$$\begin{aligned} \min_{w^{[L]}} \quad & \sum_{i=1}^{L}(e_i^{[L]})^* e_i^{[L]} \\ \text{s.t.} \quad & \mathcal{D}^L(z^{[L]}, w^{[L]}) \leqslant M. \end{aligned} \tag{1}$$

With this formulation in mind, we can begin the analysis regarding convergence of the estimates to the true nonlinearity. We summarize our principal assumptions for this analysis below.

A.1 The region over which $\mathcal{N}$ is to be identified, Ω, is an open subset of $\mathbf{R}^p$.

A.2 The LFT model structure is well-posed.

A.3 The input-output data $(u_k, y_k)_{k=1}^L$ is known, and the signal $y - \mathcal{L}_{yu}u$ is bounded.

A.4 $\mathcal{L}$ is known, $\mathcal{L}_{ye}$ is invertible with a stable inverse, and $\mathcal{L}_{yw}$ is stable.

A.5 z is measurable, there exists B such that for every $\mathcal{N} \in \mathbb{S}$,

$$\lim_{L\to\infty} \frac{1}{L}\|\mathcal{N}(z)\|_1 \leq B\|\mathcal{N}\|_1$$

A.6 There is no undermodelling. That is, there exists a true nonlinearity $\mathcal{N}^{true} \in \mathbb{S}$ and true signal $e^{true} \in \mathbb{E}$ that generated the input-output data.

A.7 $\lim_{L\to\infty} \mathcal{D}^L(z^{[L]}, \mathcal{N}^{true}(z^{[L]})) < \infty$.

A.8 e^{true} and z are independent. That is,

$$\lim_{L\to\infty} \frac{1}{L}\sum_{i=1}^{L}(e_i^{true})^*(\mathcal{L}_{ye}^{-1}\mathcal{L}_{yw}\mathcal{N}(z))_i = 0$$

uniformly over $\mathbb{S}$.

Remark 1: Assumption A.5 is related to a stationarity and density condition on z. The independence of e^{true} and z in Assumption A.8 is not defined as in the statistical sense. However, this may be satisfied by the statistical independence of e^{true} and z with further conditions on the class of nonlinearities $\mathbb{S}$.

V. Identifiability

We now address the issue of identifiability. Loosely speaking, the LFT model structure in Figure 1 is identifiable if it is possible to determine the static nonlinear block $\mathcal{N}$ uniquely on the basis of noise-free input-output experiments. Essentially, identifiability requires that no two nonlinearities in $\mathbb{S}$ result in the same input-output behavior of the LFT model structure. As is well known, identifiability concepts are of fundamental importance in system identification [13]. Let us begin with the following definition.

Definition 3: The LFT model structure in Figure 1 is <u>identifiable</u> if for every $\mathcal{N}_1 \in \mathbb{S}$, there does not exist $\mathcal{N}_1 \neq \mathcal{N}_2 \in \mathbb{S}$ such that $\mathcal{M}(\mathcal{L}, \mathcal{N}_1) = \mathcal{M}(\mathcal{L}, \mathcal{N}_2)$.

For our analysis on identifiability, we will also require the following co-measurability property. Note that this property is not pertinent towards the identification algorithm.

Definition 4: z is <u>co-measurable</u> if there exists an LTI system Ψ_C such that

$$\begin{bmatrix} \mathcal{L}_{ze} & \mathcal{L}_{zw} \end{bmatrix} = \mathcal{L}_{zu}\Psi_C.$$

Co-measurability of z implies that

$$\begin{aligned} z &= \mathcal{L}_{zu}u + \mathcal{L}_{ze}e + \mathcal{L}_{zw}w \\ &= \mathcal{L}_{zu}\begin{bmatrix} I & \Psi_C \end{bmatrix} \begin{bmatrix} u \\ e \\ w \end{bmatrix}. \end{aligned}$$

That is, all possible signals z lie in $Range(\mathcal{L}_{zu})$.

In the remainder of this section, we will develop tests for identifiability. Let the following assumptions hold for this section.

I.1 Every $\mathcal{N} \in \mathbb{S}$ is differentiable on Ω.

I.2 z is measurable and co-measurable.

I.3 For every $\mathcal{N} \in \mathbb{S}$, there exists $\omega \in \Omega$ such that $\mathcal{N}(\omega) = 0$.

The following theorem demonstrates when the identifiability can be evaulated by considering only the "forward" path from u to y. This allows us to disregard the effect of the unmeasured signal e and the feedback interconnection between z and w. For ease of notation, we will denote $\mathcal{L}_{yw}\mathcal{N}\mathcal{L}_{zu}$ to be the operator $\mathcal{L}_{yw}\mathcal{N}(\mathcal{L}_{zu}(\cdot))$.

Theorem 2: (See [4]). Let Assumptions I.1-I.3 hold. The LFT model structure is identifiable if and only if there does not exist $0 \neq \mathcal{N} \in \mathbb{S}$ such that $\mathcal{L}_{yw}\mathcal{N}\mathcal{L}_{zu} = 0$.

Theorem 2 shows that the identifiability of the LFT model structure can be determined by searching over $\mathbb{S}$ to see if $\mathcal{L}_{yw}\mathcal{N}\mathcal{L}_{zu} = 0$. For a more computationally tractable test, it can be shown that we can further simplify the test for identifiability by searching only over a set of structured matrices. For this, let us define

$$\mathbb{X} = \begin{bmatrix} X^{[1]} & & \\ & \ddots & \\ & & X^{[m]} \end{bmatrix},$$

where $X^{[i]}$ has the same input-output dimensions as $\mathcal{N}^{[i]}$. We have the following result.

Theorem 3: (See [4]). Let Assumptions I.1-I.3 hold. The LFT model structure is identifiable if and only if there does not exist $0 \neq X \in \mathbb{X}$ such that $\mathcal{L}_{yw} X \mathcal{L}_{zu} = 0$.

This result allows us to formulate a computationally simple test for identifiability. The goal is to to compute the existence of the required matrix X. Since we know that the mapping from X to $\mathcal{L}_{yw} X \mathcal{L}_{zu}$ is linear, we can determine if this mapping has a nontrivial nullspace on $\mathbb{X}$. A computable test for identifiablity is given in Figure 5.

1. Parameterize $\mathbb{X}$ as $X = \sum_i^q \theta_i K_i$, where $K_i \in \mathbf{R}^{m\times p}$ and q is the dimension of $\mathbb{X}$.
2. Perform coprime factorization of systems $\mathcal{L}_{yw} = D_{yw}^{-1} N_{yw}$ and $\mathcal{L}_{zu} = N_{zu} D_{nu}^{-1}$ with poles of N_{yw} and N_{zu} at zero.
3. Form Toeplitz matrix $\mathcal{T}$ of (finite) impulse response of $\sum_i^q \theta_i N_{yw} K_i N_{zu}$
4. Check null space of $\mathcal{T}$. If $Null(\mathcal{T}) \neq \phi$, then the LFT model structure is not identifiable.

Fig. 5. Computatable identifi ability test.

VI. Persistence of Excitation

In this section, we discuss the notion of persistently exciting signals. Intuitively, a signal is persistently exciting if it produces an input-output data set that is informative enough for a unique estimate to arise from the identification process. General conditions for persistence of excitation are very complex. However, considerable insight can be obtained in the case where the signal z is measurable.

For the remainder of this section, we will assume that the LFT model structure is identifiable. Our condition for persistence of excitation will be stated in terms of the signal z.

Definition 5: The signal z is persistently exciting if for every $\mathcal{N} \in \mathbb{S}$,

$$\lim_{L\to\infty} \frac{1}{L} \sum_{i=1}^{L} (\mathcal{L}_{ye}^{-1}\mathcal{L}_{yw}\mathcal{N}(z^{[L]}))_i^* (\mathcal{L}_{ye}^{-1}\mathcal{L}_{yw}\mathcal{N}(z^{[L]}))_i = 0$$

implies that $\|\mathcal{N}\|_1 = 0$.

Note that the persistence of excitation condition is dependent on the linear block of the LFT model structure. More specifically, a signal z that is persistently exciting for a given LFT model structure may be not persistently exciting for another. While persistence of excitation conditions are commonly independent of the model, a notion of persistence of excitation that considers the model structure may be more appropriate in the case of general interconnected systems. One should also note that if the LFT model structure is not identifiable, no persistently exciting signal exists. This explains the neccessity of the identifiability assumption throughout this section.

We now pose the question, "What types of signals are persistently exciting?" In order to present an illustrative example, let the following assumptions hold.

PP.1 Let $H = \mathcal{L}_{ye}^{-1}\mathcal{L}_{yw}$ be an FIR filter with $t+1$ taps.
PP.2 For every $\mathcal{N} \in \mathbb{S}$, there exists $\omega \in \Omega$ such that $\mathcal{N}(\omega) = 0$.

Let us begin with the following Theorem. In Example 1, we demonstrate how to construct a persistently exciting signal.

Theorem 4: (See [14]). Let $\mathcal{N} \in BV(\Omega)$. Then, there exists a sequence of functions $(\mathcal{N}_k)_{k=1}^{\infty} \subset BV(\Omega) \cap \mathcal{C}^{\infty}(\Omega)$ such that

1) $\mathcal{N}_k \to \mathcal{N}$ in $\mathcal{L}_1$
2) $\|D\mathcal{N}_k\| \to \|D\mathcal{N}\|$

Example 1: Suppose that $\mathcal{N} \in \mathcal{C}(\Omega)$. Let $\mathcal{M}(\mathcal{L},\mathcal{N})$ be identifiable. We can write the signal $\tilde{e} = H\mathcal{N}(z)$ as

$$\tilde{e}_k = h_0^*\mathcal{N}(z_k) + h_1^*\mathcal{N}(z_{k-1}) + \cdots + h_t^*\mathcal{N}(z_{k-t}).$$

Define 0_t to be a sequence of t zeros. We will construct the signal z as follows. Let

$$z = (0_t, a_1, 0_t, a_2, 0_t, a_3, 0_t, \ldots),$$

where $(a_k)_{k=1}^{\infty}$ is dense in Ω. Then,

$$\tilde{e} = (\ldots, h_0^*\mathcal{N}(a_1), h_1^*\mathcal{N}(a_1), \ldots, \\ h_t^*\mathcal{N}(a_1), h_0^*\mathcal{N}(a_2), h_1^*\mathcal{N}(a_2), \ldots).$$

Suppose that $\lim_{L\to\infty} \frac{1}{L}\sum_{i=1}^{L}(H\mathcal{N}(z^{[L]}))_i^*(H\mathcal{N}(z^{[L]}))_i = 0$. Then for every $\epsilon > 0$, there exists I such that for almost every $i > I$, $\|h_j\mathcal{N}(a_i)\| < \epsilon$ for $j = 0, \ldots, t$. Since $\mathcal{N}$ is continuous and $(a_i)_{i=1}^{\infty}$ is dense in Ω, this implies that $h_j\mathcal{N} = 0$ on Ω for $j = 0, \ldots, t$. We can then conclude that $H\mathcal{N} = 0$ on Ω. The identifiability of the LFT model structure results in $\mathcal{N} = 0$. As a result, the signal z is persistently exciting.

Now suppose that $\mathcal{N} \in BV(\Omega)$. From Theorem 4, there exists a sequence $(\mathcal{N}_k) \subset BV(\Omega) \cap \mathcal{C}^{\infty}(\Omega)$ such that

1) $\mathcal{N}_k \to \mathcal{N}$ in L_1
2) $\|D\mathcal{N}_k\| \to \|D\mathcal{N}\|$

By the triangle inequality,

$$\|\mathcal{N}\|_1 \leq \|\mathcal{N}_k - \mathcal{N}\|_1 + \|\mathcal{N}_k\|_1.$$

We have from Assumption A.5 that as a functional of $\mathcal{N}$, $\lim_{L\to\infty} \frac{1}{L}\sum_{i=1}^{L}(H\mathcal{N}(z^{[L]}))_i^*(H\mathcal{N}(z^{[L]}))_i$ is continuous in L_1 . Then, since z is persistently exciting for every $\mathcal{N}_k$ and $\|\mathcal{N}_k - \mathcal{N}\|_1 \to 0$, z is persistently exciting for $\mathcal{N}$.

Example 2: Example 1 demonstrates how a dense, impulsive input can be persistently exciting. Despite the fact that we are seeking a nonparametric estimate, some very simple signals may also be persistently exciting. Consider the LFT model structure with the following linear and nonlinear blocks.

$$\mathcal{L} = \begin{bmatrix} 1 & z^{-2} \end{bmatrix}$$
$$\mathcal{N} = \begin{bmatrix} f \\ g \end{bmatrix},$$

where f and g are static nonlinearities and z^{-2} is a two-sample delay operator. Then, a sinusoidal input with frequency ω chosen such that $\frac{\omega}{\pi}$ is irrational is persistently exciting.

The key property of a persistently exciting signal is that it satisfies a density in both space and time.

VII. Convergence Results

We now present our results regarding convergence of the estimated nonlinear function. It is important to note that the proof for convergence rests heavily on Assumption A.8.

Let us assume that $\mathcal{D}^L(z^{[L]}, \mathcal{N}^{true}(z^{[L]})) \leq M$, which implies that $\|D\mathcal{N}^{true}\| \leq \sqrt{M}$. We can now state our main result.

Theorem 5: Let the LFT model structure be identifiable and let z be persistently exciting. Let $\hat{\mathcal{I}}^{[L]}$ be the estimated nonlinearity, formed from an interpolation of the points $(z^{[L]}, \hat{w}^{[L]})$. Then,

$$\|\hat{\mathcal{I}}^{[L]} - \mathcal{N}^{true}\|_1 \longrightarrow 0.$$

Proof: Let $H = \mathcal{L}_{ye}^{-1}\mathcal{L}_{yw}$. Note that the cost function can be written as

$$\begin{aligned} J^L(\mathcal{N}(z^{[L]})) &= \frac{1}{L}\sum_{i=1}^{L}(e_i^{[L]})^* e_i^{[L]} \\ &= \frac{1}{L}\sum_{i=1}^{L}(e^{true})_i^*(e^{true})_i \\ &\quad + \frac{2}{L}\sum_{i=1}^{L}(e^{true})_i^*(H\tilde{\mathcal{N}}(z^{[L]}))_i \\ &\quad + \frac{1}{L}\sum_{i=1}^{L}(H\tilde{\mathcal{N}}(z^{[L]}))_i^*(H\tilde{\mathcal{N}}(z^{[L]}))_i, \end{aligned}$$

where $\tilde{\mathcal{N}} = \mathcal{N}^{true} - \hat{\mathcal{I}}^{[L]}$. From Lemma 1 , $\hat{\mathcal{I}}^{[L]} \in \mathbb{S}$. Thus, we have by Assumption A.8 that the middle term converges to zero. It then follows that

$$J^L(\hat{\mathcal{I}}^{[L]}(z^{[L]})) \leq J^L(\mathcal{N}^{true}(z^{[L]}))$$
$$\limsup_{L\to\infty} J^L(\hat{\mathcal{I}}^{[L]}(z^{[L]})) \leq \limsup_{L\to\infty} J^L(\mathcal{N}^{true}(z^{[L]})).$$

Since

$$\limsup_{L\to\infty} J^L(\mathcal{N}^{true}(z^{[L]})) = \limsup_{L\to\infty} \frac{1}{L}\sum_{i=1}^{L}(e_i^{[L]})^* e_i^{[L]},$$

we have

$$\lim_{L\to\infty} \frac{1}{L}\sum_{i=1}^{L}(H\tilde{\mathcal{N}}(z^{[L]}))_i^*(H\tilde{\mathcal{N}}(z^{[L]}))_i = 0.$$

Since z is persistently exciting, this implies that

$$\|\hat{\mathcal{I}}^{[L]} - \mathcal{N}^{true}\|_1 \longrightarrow 0.$$

■

VIII. Conclusion

An algorithm for the identification of static nonlinear maps in interconnected systems was proposed. Utilizing the dispersion function, the identification problem is reduced to a least squares problem. Formal notions of identifiability and persistence of excitation have been developed, and convergence of the estimates to the true nonlinearity has been shown. Future work may involve a thorough investigation of the persistence of excitation condition and a more delicate treatment of Assumption A.8.

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