Robust H_2 and H_∞ Filters for Uncertain LFT Systems

Kunpeng Sun and Andy Packard
Department of Mechanical Engineering
University of California, Berkeley, CA, 94706
{kpsun, pack}@jagger.me.berkeley.edu

Abstract

In this paper, the robust H_2 and H_∞ filter design problems are considered, where the uncertainties, unstructured or structured, are norm bounded and represented by linear fractional transformation (LFT). The main result is that after upper-bounding the objectives, the problems of minimizing the upper bounds are converted to finite dimensional convex optimization problems involving linear matrix inequalities (LMIs). These are extensions of the results for systems with polytopic uncertainty. It is also shown that for the unstructured, norm bounded uncertainty case, the results here are less conservative than former results, where Riccati equation approach are used. A numerical example is given to illustrate the results.

1 Introduction

The problem of state estimation is fundamental to control theory and signal processing, and several approaches have been developed, for instance, H_2 optimal filtering, H_∞ optimal filtering [12] and set-membership approach [1]. In recent years, uncertainties in the dynamic models are taken into account, and numerous papers on the robust filter have appeared, namely, [8, 19] and references therein, and the text by I.R. Petersen and A.V. Savkin [18] is a comprehensive collection of Riccati based (H_2, H_∞ and set-membership) approaches.

The robust H_2 filtering problem is often formulated in a Kalman-like stochastic context, where the uncertain dynamic system is subjected to white Gaussian disturbance. The objective of the design problem is to find the filter parameters such that the worst case mean square estimation error is minimized. To our knowledge, this is done by first over-bounding the objective, and then developing techniques to minimize the bound. Petersen and McFarlane [16, 17], Xie et. al [27], Theodor and Shaked [23, 21] consider this problem with parameter uncertainties entering the system affinely and lie in an unstructured, norm bounded set. A Riccati equation based approach is used to minimize the upper bound. de Souza [3], Geromel [10] and Palhares and Peres [15] consider the case that uncertainty is in a polytope. They convert the problem of minimizing the upper bound to a convex optimization problem involving LMIs. Fu et al. consider the finite horizon robust H_2 filtering problem in [8]. They put a linear structure on the filter and get a time varying filter.

In the robust H_∞ filtering problem, the filter is designed such that the worst-case induced L_2 gain from process noise to estimation error is minimized. Similar to the H_2 filtering problem, it's a common practice that an upper bound is derived first, and then the bound is minimized based on approaches such as Riccati equations and LMIs. Xie, de Souza and Fu, [25, 26, 7] consider the robust H_∞ filtering problem for norm bounded, unstructured uncertain dynamic systems using Riccati equation approach. Palhares and Peres [14], Geromel et al. [9, 10] use LMI approach to tackle the uncertain dynamic systems with polytopic uncertainties. Li and Fu [11] consider the H_∞ filtering problem for systems with IQC constraints. They formulate the problem using matrix inequalities, but for the general problem, it's not convex.

The robust estimation problem is closely related to the robust control problem. In [5, 28, 20, 29], and references therein, the robust H_2 and H_∞ output feedback controller design problems for uncertain LFT systems are considered. The design problem via LMIs turned out to be bilinear, hence not convex. Some heuristics are used to solve the problem.

In this paper, we consider the robust H_2 and H_∞ filtering problem for dynamic systems with time varying LFT uncertainty, which is norm bounded, unstructured or structured. Both design objectives can be upper bounded, based on a single quadratic Lyapunov function for each problem. The problems are then reformulated as minimizing these upper bounds. By a nonlinear transformation of variables [3, 10], both problems are converted to finite dimensional convex optimization problems involving LMIs.

The contribution of this paper is the treatment of norm bounded (both structured and unstructured) LFT uncertainty using LMI (rather than Riccati) methods. In the norm bounded unstructured uncertainty case, we establish necessary and sufficient LMI conditions for finding the upper bounds, which are less conservative than those methods based on Riccati equations [18, 25].

The remainder of this paper is organized as follows. In section 2, the problem description and some preliminary results are presented. In section 3, we reduce the problem to finite dimensional optimization involving LMIs, and then reduced further to one with fewer design variables. The H_∞ filtering problem is considered in section 4. An example is given in section 5, and conclusions are drawn in section 6.

The notation in this paper is fairly standard. $\mathbb{R}^{m\times n}$ is the set of real $m \times n$ matrices. For $M \in \mathbb{R}^{n\times n}$, $M > 0$ ($M \geq 0$) indicates that M is positive definite (semidefinite), and $M < 0$ means that it is negative definite. We use $\mathcal{S}_+^n$ to denote the set of positive definite matrices. $tr(\cdot)$ stands for the matrix trace. $\mathcal{E}\{\cdot\}$ denotes the expectation operator. For $U \in \mathbb{R}^{n\times p}$, $rank(U) = p$ ($p \leq n$), $U_\perp$ denotes any orthogonal complement of U, defined as $U_\perp^T U = 0$, and $rank([U_\perp\ U]) = n$.

2 Problem Setup and Preliminaries

Consider the following class of uncertain continuous-time systems

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = M(\Delta(t)) \begin{bmatrix} x(t) \\ w(t) \\ v(t) \end{bmatrix} \tag{1}$$

where $x(0) = x_0$, and $M(\Delta(t))$ is given by

$$\begin{aligned} M(\Delta(t)) \quad = \quad & \begin{bmatrix} A & B_w & 0 \\ C & 0 & B_v \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \Delta(t) \\ & \times (I - H\Delta(t))^{-1} \begin{bmatrix} R_1 & 0 & 0 \end{bmatrix} \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ are the states, $w(t) \in \mathbb{R}^{n_w}$ is the process noise, $y(t) \in \mathbb{R}^m$ are the measurements, and $v(t) \in \mathbb{R}^{n_v}$ is the measurement noise. A, C, B_w and B_v are known constant matrices with appropriate dimensions. The uncertainty matrix $\Delta(t)$ is in general time-varying, unstructured or structured, and satisfying a given norm bound. We use $\mathbf{\Delta_u} := \{\Delta \in \mathbb{R}^{n_p\times n_q} : \|\Delta\| \leq 1\}$ to denote the unstructured uncertainties. The structured uncertainties is denoted by $\mathbf{\Delta_s} := \{\Delta = \mathrm{diag}(\delta_1 I_{q_1}, \ldots, \delta_l I_{q_l}, \Delta_{l+1}, \ldots, \Delta_{l+f}) : \|\Delta\| \leq 1,\ \delta_i \in \mathbb{R},\ \Delta_i \in \mathbb{R}^{q_i\times q_i}$ and $\sum_{i=1}^{l+f} q_i = n_p = n_q\}$. For convenience, we use $\mathbf{\Delta}$ to denote both cases. L_1, L_2, R_1 and H are known constant matrices with appropriate dimensions, which specify how the elements of the nominal matrices A and C are affected by the uncertain parameter $\Delta(t) \in \mathbf{\Delta}$.

This linear fractional transformation (LFT) representation of uncertainty has great generality and is widely used in robust control theory, for instance [4, 13, 29]. This framework includes the case when parameters perturb each coefficient of the data matrices in a polynomial or rational manner ([4]). In this paper, we assume the representation (1) is well-posed over $\mathbf{\Delta}$, meaning that $\det(I - H\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}$. Under this assumption, the uncertain part can be isolated from known part and written equivalently as follows,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_w w(t) + L_1 p(t) \\ y(t) &= Cx(t) + B_v v(t) + L_2 p(t) \\ q(t) &= R_1 x(t) + Hp(t) \\ p(t) &= \Delta(t) q(t), \qquad \Delta(t) \in \mathbf{\Delta} \end{aligned}$$

where $p(t) \in \mathbb{R}^{n_p}$ and $q(t) \in \mathbb{R}^{n_q}$ are perturbation signals.

The objective is to design a filter to estimate $z(t) := Lx(t)$, where $L \in \mathbb{R}^{r\times n}$. Restrict the estimator to be full order with linear structure:

$$\dot{\hat{x}}(t) = A_f \hat{x}(t) + B_f y(t) \tag{3}$$

$$\hat{z}(t) = L_f \hat{x}(t) \tag{4}$$

where $A_f \in \mathbb{R}^{n\times n}$, $B_f \in \mathbb{R}^{n\times m}$ and $L_f \in \mathbb{R}^{r\times n}$ are constant matrices. Define the estimation error by $e(t) := z(t) - \hat{z}(t)$. Let $\eta(t) := [x(t)^T\ \hat{x}(t)^T]^T$ and $\xi(t) \in \mathbb{R}^{n_w+n_v}$ to denote the states and noise signal, respectively, of the augmented system, which is given by

$$\dot{\eta}(t) = [\bar{A} + \bar{L}\Delta(t)(I - H\Delta(t))^{-1}\bar{E}]\eta(t) + \bar{B}\xi(t) \tag{5}$$

$$e(t) = \bar{C}\eta(t) \tag{6}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix},\ \bar{L} = \begin{bmatrix} L_1 \\ B_f L_2 \end{bmatrix},\ \bar{E} = \begin{bmatrix} R_1 & 0 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B_w & 0 \\ 0 & B_f B_v \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} L & -L_f \end{bmatrix}$$

In this paper, two problems will be considered.

$\mathbf{H_2}$ filtering problem: similar to the Kalman filter, we take a stochastic interpretation. In equation (1), assume $w(t)$ and $v(t)$ are zero mean white Gaussian process, with $\mathcal{E}[w(t)w(l)^T] = \delta(t-l)I_{n_w}$, $\mathcal{E}[v(t)v(t)^T] = \delta(t-l)I_{n_v}$ and $\mathcal{E}[w(t)v(l)^T] = 0$, where $\delta(t)$ is the Kronecker delta. Hence in equation (5), $\xi(t)$ is a zero-mean white noise signal satisfying $\mathcal{E}[\xi(t)\xi(l)^T] = \delta(t-l)I_{n_w+n_v}$. The H_2 performance objective is $\sigma := \lim_{T\to\infty} \sigma_T$, where $\sigma_T = \sup_{\Delta(\cdot)\in\mathbf{\Delta}} \mathcal{E}\left\{\frac{1}{T}\int_0^T e^T(t)e(t)\,dt\right\}$. The filter design objective is to minimize σ:

$$\min_{A_f, B_f, L_f} \quad \sigma \tag{7}$$

$$\text{Subject to} \quad (5) \text{ and } (6) \tag{8}$$

$\mathbf{H}_\infty$ filtering problem: For $\xi \in L_2[0,\infty]$, define the induced L_2 operator norm to be $\rho := \sup_{\Delta(t)\in\mathbf{\Delta}} \sup_{\|\xi\|_2\neq 0} \frac{\|e\|_2}{\|\xi\|_2}$. The filter design objective is to minimize ρ:

$$\min_{A_f,B_f,L_f} \quad \rho \tag{9}$$

$$\text{Subject to} \quad (5) \text{ and } (6) \tag{10}$$

In this paper, we assume that system (1) is quadratically stable. This is a typical assumption of all work in this area. Before solving the problem, we shall need the following lemmas.

Lemma 1 (Boyd, [2]) *Let $G \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times p}$, $V \in \mathbb{R}^{n\times q}$ and $p, q < n$. There exists a matrix $X \in \mathbb{R}^{p\times q}$ such that $G + UXV^T + VX^TU^T > 0$ if and only if $U_\perp^T G U_\perp > 0$ and $V_\perp^T G V_\perp > 0$.*

The following lemmas can be derived from $\mathcal{S}$-procedure. Notice that in lemma 2, the result is a *necessary and sufficient* condition.

Lemma 2 (El Ghaoui, [5]) *Let $T_1 = T_1^T$, T_2, T_3, T_4 be real matrices of appropriate size. We have $\det(I - T_4\Delta) \neq 0$ and $T_1 + T_2\Delta(I-T_4\Delta)^{-1}T_3 + T_3^T(I-T_4\Delta)^{-T}\Delta^T T_2^T < 0$ for every $\Delta \in \mathbf{\Delta_u}$, if and only if there exist a scalar $\lambda > 0$ such that*

$$\begin{bmatrix} T_1 + \lambda T_3^T T_3 & T_2 + \lambda T_3^T T_4 \\ T_2^T + \lambda T_4^T T_3 & \lambda(T_4^T T_4 - I) \end{bmatrix} < 0$$

The next lemma is the result for structured uncertainty $\Delta \in \mathbf{\Delta_s}$. It's a generalization of above lemma, we omit the proof here. For convenience, define the following structured subspace $\mathbf{S}$ and $\mathbf{G}$: $\mathbf{S} := \{\mathrm{diag}(S_1,\dots,S_l,\mu_1 I_{q_{l+1}},\dots,\mu_s I_{q_{l+f}}) : S_i = S_i^T \in \mathbb{R}^{q_i\times q_i},\ i = 1,\dots,l\}$, and $\mathbf{G} := \{\mathrm{diag}(G_1,\dots,G_l,0_{q_{l+1}},\dots,0_{q_{l+f}}) : G_i = -G_i^T \in \mathbb{R}^{q_i\times q_i},\ i = 1,\dots,l\}$.

Lemma 3 *Let $T_1 = T_1^T$, T_2, T_3, T_4 be real matrices of appropriate size. We have $\det(I - T_4\Delta) \neq 0$ and $T_1 + T_2\Delta(I - T_4\Delta)^{-1}T_3 + T_3^T(I - T_4\Delta)^{-T}\Delta^T T_2^T < 0$ for every $\Delta \in \mathbf{\Delta_s}$, if there exist block-diagonal matrices $S \in \mathbf{S}$ and $G \in \mathbf{G}$ such that $S > 0$ and*

$$\begin{bmatrix} T_1 + T_3^T S T_3 & T_2 + T_3^T S T_4 + T_3^T G \\ T_2^T + T_4^T S T_3 - G T_3 & T_4^T S T_4 + T_4^T G - G T_4 - S \end{bmatrix} < 0.$$

3 Robust H_2 Filtering

3.1 An Upper Bound

The robust H_2 filtering problem (7) is hard to solve directly, but we can find an upper bound for σ, and the new objective is to minimize this upper bound. Define a set $\mathcal{X}_2 := \{P \in \mathcal{S}_+^{2n} : \bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T\bar{C} < 0,\ \forall\Delta \in \mathbf{\Delta}\}$, where $\bar{A}_\Delta = \bar{A} + \bar{L}\Delta(I - H\Delta)^{-1}\bar{E}$. The next lemma shows that an upper bound for the performance measure σ can be derived from a single quadratic Lyapunov function [2, 24].

Lemma 4 (F. Wu, [24]) *Since $\mathbf{\Delta}$ is compact, the linear uncertain system (5) is quadratically stable if and only if $\mathcal{X}_2$ is nonempty, furthermore, $\sigma \leq \inf_{P\in\mathcal{X}_2} tr(\bar{B}^T P\bar{B}) =: \alpha$. If the system is not uncertain and linear time-invariant, we have $\sigma = \alpha$.*

The goal now is to design parameters A_f, B_f, L_f and $P \in \mathbb{R}^{2n\times 2n}$ to minimize the upper bound α of the worst case performance measure σ:

$$\min_{P,A_f,B_f,L_f} \quad tr(\bar{B}^T P\bar{B}), \tag{11}$$

$$\text{Subject to} \quad \bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T\bar{C} < 0,\ \forall\Delta \in \mathbf{\Delta} \tag{12}$$

$$P > 0 \tag{13}$$

This problem formulation is also called *guaranteed cost state estimation* in [16, 18], where $L = I$ and $H = 0$. From lemma 4, problem (11)-(13) is feasible if and only if the set $\mathcal{X}_2$ is nonempty, and a necessary condition for this is the assumption that system (1) is quadratically stable.

3.2 Robust Filter Synthesis via LMIs

The inequality constraints of problem (11)-(13) are not convex in P, A_f, B_f and L_f. In the next theorems, we use a recently discovered nonlinear transformation ([3, 10]) to convert problem (11)-(13) to a convex one. In theorem 5, we state the result for dynamic systems with unstructured uncertainty. The result for systems with structured uncertainty is given in theorem 9.

Theorem 5 *When the uncertainties are unstructured, i.e., $\Delta(\cdot) \in \mathbf{\Delta_u}$, for a given number $\gamma > 0$, $tr(\bar{B}^T P\bar{B}) < \gamma$ and (12), (13) are satisfied if and only if the following LMI problem in M_A, P_0, $P_1 \in \mathbb{R}^{n\times n}$, $M_B \in \mathbb{R}^{n\times m}$, $M_L \in \mathbb{R}^{r\times n}$, $N \in \mathbb{R}^{(n_w+n_v)\times(n_w+n_v)}$ and $\lambda \in \mathbb{R}$ is feasible:*

$$tr(N) < \gamma,\ P_1 - P_0 > 0,\ P_0 > 0,\ \lambda > 0, \tag{14}$$

$$\left[\begin{array}{c|c|c|c} M_1 & M_2 & M_3 & L^T \\ \hline M_2^T & M_A + M_A^T & P_0 L_1 + M_B L_2 & -M_L^T \\ \hline M_3^T & L_1^T P_0 + L_2^T M_B^T & \lambda(H^T H - I_{n_p}) & 0 \\ \hline L & -M_L & 0 & -I_r \end{array}\right] < 0 \tag{15}$$

$$\begin{bmatrix} N_1 & N_3 & B_w^T P_1 & B_w^T P_0 \\ N_3^T & N_2 & B_v^T M_B^T & B_v^T M_B^T \\ P_1 B_w & M_B B_v & P_1 & P_0 \\ P_0 B_w & M_B B_v & P_0 & P_0 \end{bmatrix} \geq 0 \tag{16}$$

where in (15), M_1, M_2 and M_3 are defined as follows:

$$\begin{aligned} M_1 &= A^T P_1 + P_1 A + M_B C + C^T M_B^T + \lambda R_1^T R_1, \\ M_2 &= A^T P_0 + M_A + C^T M_B^T, \\ M_3 &= P_1 L_1 + M_B L_2 + \lambda R_1^T H, \end{aligned}$$

and in (16), N is partitioned as $\begin{bmatrix} N_1 & N_3 \\ N_3^T & N_2 \end{bmatrix}$ and $N_1 \in \mathbb{R}^{n_w \times n_w}$.

With the solution N, P_0, P_1, M_A, M_B and M_L found, the matrices of the filter are given by $A_f = P_3^{-1} M_A (P_3^T)^{-1} P_2$, $B_f = P_3^{-1} M_B$, $L_f = M_L (P_3^T)^{-1} P_2$, where P_2 and P_3 are any $n \times n$ matrices with P_2 symmetric and $P_3 P_2^{-1} P_3^T = P_0$. Moreover, the asymptotic mean square estimation error satisfies $\sigma \le tr(N)$ for all admissible uncertainties.

Proof: Partition P as $P = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix}$, where $P_1 \in \mathbb{R}^{n \times n}$. Note that, without loss of generality, we can assume P_3 is nonsingular, (see [3, pp. 179]). We introduce new variables, let $P_0 := P_3 P_2^{-1} P_3^T$, $M_A := P_3 A_f P_2^{-1} P_3^T$, $M_B := P_3 B_f$ and $M_L := L_f P_2^{-1} P_3^T$.

First, using Schur complement, $tr(\bar{B}^T P \bar{B}) < \gamma$ if and only if there exists $N \in \mathbb{R}^{(n_w+n_v)\times(n_w+n_v)}$ such that $tr(N) < \gamma$ and $\begin{bmatrix} N & \bar{B}^T P \\ P\bar{B} & P \end{bmatrix} \ge 0$. This is equivalent to

$$\begin{bmatrix} N_1 & N_3 & B_w^T P_1 & B_w^T P_3 \\ N_3^T & N_4 & B_v^T M_B^T & B_v^T B_f^T P_2 \\ P_1 B_w & M_B B_v & P_1 & P_3 \\ P_3^T B_w & P_2 B_f B_v & P_3^T & P_2 \end{bmatrix} \ge 0 \quad (17)$$

Let $J_1 = \mathrm{diag}\{I_n, I_m, I_n, P_2^{-1} P_3^T\}$, and multiply (17) by J_1^T and J_1 from left and right, respectively, we obtain (17) is true if and only if condition (16) is satisfied.

Second, write the constraint (12) explicitly,

$$\begin{aligned} &[\bar{A} + \bar{L}\Delta(I - H\Delta)^{-1}\bar{E}]^T P + \\ &\quad P[\bar{A} + \bar{L}\Delta(I - H\Delta)^{-1}\bar{E}] + \bar{C}^T \bar{C} < 0 \quad (18) \end{aligned}$$

By lemma 2, (18) is true for all $\Delta \in \mathbf{\Delta_u}$ if and only if $\exists\, \lambda > 0$, such that

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \bar{C}^T \bar{C} + \lambda \bar{E}^T \bar{E} & P\bar{L} + \lambda \bar{E}^T H \\ \bar{L}^T P + \lambda H^T \bar{E} & \lambda(H^T H - I_{n_p}) \end{bmatrix} < 0 \quad (19)$$

Let $J = \mathrm{diag}\{I, P_2^{-1} P_3^T, I\}$, and multiply (19) by J^T and J from left and right, respectively, we obtain the equivalent LMI condition:

$$\left[\begin{array}{c|c|c} M_1 + L^T L & M_2 - L^T M_L & M_3 \\ \hline M_2^T - M_L^T L & M_A + M_A^T + M_L^T M_L & P_0 L_1 + M_B L_2 \\ \hline M_3^T & L_1^T P_0 + L_2^T M_B^T & \lambda(H^T H - I_{n_p}) \end{array}\right] < 0 \quad (20)$$

This equation is not an LMI yet because of the term $M_L^T M_L$. Using Schur complement, we can get the equivalent condition (15).

Last, by Schur complement, the constraint (13) is true if and only if $P_2 > 0$ and $P_1 - P_3 P_2 P_3^T = P_1 - P_0 > 0$. Since P_3 is nonsingular, $P_2 > 0$ if and only if $P_3 P_2 P_3^T = P_0 > 0$. This is condition (14). $\sigma \le tr(N)$ is the conclusion of lemma 4. Thus the proof is complete. ∎

Remark 6 *Using theorem 5, the design problem (11)-(13) can be transformed to an equivalent one, which is convex and can be solved efficiently:*

$$\min \quad \gamma \quad (21)$$

$$\text{Subject to} \quad (14),\ (15) \text{ and } (16) \quad (22)$$

Remark 7 *When $H = 0$ in (2) and $L = I$, the problem formulation (11)-(13) is the same as in [16, 27, 21], etc., where a particular structure, parameterized by a scalar, is imposed on the solution. The upper bound is then minimized over the scalar. In contrast, here we are solving the optimization problem directly, with the help of lemma 2. Since the result in theorem 5 is* necessary and sufficient, *the upper bound here must be less than or equal to that obtained in [16, 27, 21], with more required computation as a trade off.*

Remark 8 (C. de Souza, [3]) *The matrix P_3 can be viewed as a similarity transformation on the state-space realization of the filter and has no effect on input-output property of the filter. Thus, we can set $P_3 = I_n$, and suitable state-space matrices of the optimal filter are: $A_f = M_A P_0^{-1}$, $B_f = M_B$, $L_f = M_L P_0^{-1}$. Furthermore, when there is no uncertainty, the robust filter reduces to the standard Kalman filter.*

In practice, the uncertainty is often structured, for example, uncertainty with block diagonal structure arises naturally in the interconnected systems. The design would be less conservative to take this structure into account, than using the unstructured result directly. In this case, the robust H_2 filtering problem can be formulated to LMI conditions similarly by using lemma 3. We omit the proof for simplicity.

Theorem 9 *When the uncertainties are structured, i.e., $\Delta(\cdot) \in \mathbf{\Delta_s}$, problem (11)-(13) can be obtained in terms of the following LMI problem in M_A, P_0, $P_1 \in \mathbb{R}^{n\times n}$, $M_B \in \mathbb{R}^{n\times m}$, $M_L \in \mathbb{R}^{r\times n}$, $N \in \mathbb{R}^{(n_w+n_v)\times(n_w+n_v)}$ and $S \in \mathbf{S}, G \in \mathbf{G}$:*

$$\min_{N,P_0,P_1,M_A,M_B,M_L,S,G} \quad tr(N), \quad (23)$$

$$\text{Subject to } P_1 - P_0 > 0,\ P_0 > 0,\ S > 0,\ (16) \quad (24)$$

$$\left[\begin{array}{c|c|c|c} M_4 & M_2 & M_5 & L^T \\ \hline M_2^T & M_A + M_A^T & P_0 L_1 + M_B L_2 & -M_L^T \\ \hline M_5^T & \begin{array}{c} L_1^T P_0 + \\ L_2^T M_B^T \end{array} & \begin{array}{c} H^T S H + H^T G \\ -GH - S \end{array} & 0 \\ \hline L & -M_L & 0 & -I_r \end{array}\right] < 0, \tag{25}$$

where M_2 is given as before and

$$\begin{aligned} M_4 &= A^T P_1 + P_1 A + M_B C + C^T M_B^T + R_1^T S R_1, \\ M_5 &= P_1 L_1 + M_B L_2 + R_1^T S H + R_1^T G. \end{aligned}$$

With the optimal solution N, P_0, P_1, M_A, M_B and M_L found, the matrices of the filter are given as before. Moreover, the asymptotic mean square estimation error satisfies $\sigma \leq tr(N)$ for all admissible uncertainties.

3.3 Elimination of Filter Parameters

The number of design variables in above optimization problems can be made smaller without introducing any conservatism. In fact, variable M_A can be eliminated from theorem 5 and 9, and the alternative formulations are still convex optimization problems involving LMIs. Here we only give the result for the unstructured uncertainty case — the formula for the structured one can be obtained similarly.

Theorem 10 *When the uncertainties are unstructured, i.e., $\Delta(\cdot) \in \mathbf{\Delta_u}$, for a given number $\gamma > 0$, $tr(\bar{B}^T P \bar{B}) < \gamma$ and (12), (13) are satisfied if and only if the following LMI problem in P_0, $P_1 \in \mathbb{R}^{n\times n}$, $M_B \in \mathbb{R}^{n\times m}$, $M_L \in \mathbb{R}^{r\times n}$, $N \in \mathbb{R}^{(n_w+n_v)\times(n_w+n_v)}$ and $\lambda \in \mathbb{R}$ is feasible:*

$$(14),\ (16) \text{ and} \tag{26}$$

$$\left[\begin{array}{c|c} M_1 + L^T L & M_3 \\ \hline M_3^T & \lambda(H^T H - I_{n_p}) \end{array}\right] < 0 \tag{27}$$

$$\left[\begin{array}{c|c|c} M_6 & \begin{array}{c} (P_1 - P_0)L_1 + \\ \lambda R_1^T H \end{array} & M_L^T \\ \hline \begin{array}{c} L_1^T (P_1 - P_0) + \\ \lambda H^T R_1 \end{array} & \lambda(H^T H - I_{n_p}) & 0 \\ \hline M_L & 0 & -I_r \end{array}\right] < 0 \tag{28}$$

where

$$\begin{aligned} M_6 := {} & A^T (P_1 - P_0) + (P_1 - P_0)A + L^T L + \\ & M_L^T L + L^T M_L + \lambda R_1^T R_1 \end{aligned}$$

After getting N, P_0, P_1, M_B, M_L and λ, find M_A such that (20) holds. This is an LMI feasibility problem only in M_A, which is guaranteed to be feasible.

With the optimal solution N, P_0, P_1, M_A, M_B and M_L found, the matrices of the filter are given as before. Moreover, the asymptotic mean square estimation error satisfies $\sigma \leq tr(N)$ for all admissible uncertainties.

Proof: Notice that we only eliminate M_A from inequality (15), all other conditions and results in theorem 5 are intact. For convenience, we consider (20) instead of (15). Let $U^T = [I\ I\ 0_{n_p\times n}]$, $V^T = [0\ I\ 0_{n_p\times n}]$ and $R :=$

$$\left[\begin{array}{c|c|c} M_1 + L^T L & \begin{array}{c} A^T P_0 + C^T M_B^T \\ -L^T M_L \end{array} & M_3 \\ \hline \begin{array}{c} P_0 A + M_B C \\ -M_L^T L \end{array} & M_L^T M_L & \begin{array}{c} P_0 L_1 + \\ M_B L_2 \end{array} \\ \hline M_3^T & L_1^T P_0 + L_2^T M_B^T & \begin{array}{c} \lambda(H^T H \\ -I_{n_p}) \end{array} \end{array}\right].$$

Then equation (20) can be written as

$$R + U M_A V^T + V M_A^T U^T < 0 \tag{29}$$

By lemma 1, equation (29) is true if and only if (27) is true and

$$\left[\begin{array}{c|c} Q + M_L^T M_L & \begin{array}{c} (P_1 - P_0)L_1 \\ +\lambda R_1^T H \end{array} \\ \hline \begin{array}{c} L_1^T (P_1 - P_0) \\ +\lambda H^T R_1 \end{array} & \lambda(H^T H - I_{n_p}) \end{array}\right] < 0 \tag{30}$$

Equation (30) is not an LMI yet because of the term $M_L^T M_L$. Using Schur complement, one can get the equivalent LMI condition (28).

After solving above problem, find M_A such that (29) holds. By the elimination lemma 1, this feasibility problem is guaranteed to be feasible. ■

4 Robust H_∞ Filtering

In this section, the robust H_∞ filtering problem is considered. All the results for robust H_2 filtering have counterparts for robust H_∞ filtering. Here we only give the result for systems with unstructured uncertainty, other results can be obtained similarly. As before, an upper bound for the induced L_2 gain of the uncertain system will be given first, and the problem is reformulated using this upper bound.

Lemma 11 (Boyd, [2]) *For system (5) and (6), if there exists a quadratic function $V(x) = x^T P x$, $P > 0$ and $\gamma \geq 0$, such that for all t and all admissible x and ξ,*

$$\frac{d}{dt} V(x) + e^T e - \gamma^2 \xi^T \xi \leq 0 \quad \forall \Delta(t) \in \mathbf{\Delta}, \tag{31}$$

then the induced L_2 gain for this system is less than γ.

We can express this result by a matrix inequality. In fact, when there is no uncertainty and the system is linear time invariant, the following lemma reduce to the well known bounded real lemma.

Lemma 12 *For system (5) and (6), if there exists $\gamma \geq 0$ and $P > 0$ such that*

$$\begin{bmatrix} \bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T\bar{C} & P\bar{B} \\ \bar{B}^T P & -\gamma^2 I \end{bmatrix} \leq 0 \quad \forall \Delta \in \mathbf{\Delta}, \quad (32)$$

then the induced L_2 gain for this system is less than γ.

Proof: By lemma 11, we only need to show condition (32) is sufficient for (31). To this end, we multiply (32) by $[x^T \xi^T]$ and $[x^T \xi^T]^T$ from left and right, respectively, and get that $\forall x$, ξ and $\Delta \in \mathbf{\Delta}$, we have

$$\begin{bmatrix} x^T & \xi^T \end{bmatrix} \begin{bmatrix} \bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T\bar{C} & P\bar{B} \\ \bar{B}^T P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \leq 0$$

This can be written as

$$x^T(\bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T\bar{C})x + \xi^T\bar{B}^T P x + x^T P\bar{B}\xi - \gamma^2\xi^T\xi \leq 0$$

Combined with the dynamic system equations (5) and (6), we know this implies condition (31). This completes the proof. ■

The problem now is to minimize this upper bound γ by choosing the filter parameters, A_f, B_f, L_f, and the Lyapunov variable P. Similar to theorem 5, by a change of variables and the result of lemma 1, we can convert the problem to a finite dimensional one, involving LMIs. The proof is similar with that of theorem 5, hence omitted here.

Theorem 13 *Consider system (5) and (6), when the uncertainties are unstructured, i.e., $\Delta(\cdot) \in \mathbf{\Delta_u}$, then for a given number $\gamma > 0$, (32) and $P > 0$ are satisfied if and only if the following LMI problem in M_A, P_0, $P_1 \in \mathbb{R}^{n\times n}$, $M_B \in \mathbb{R}^{n\times m}$, $M_L \in \mathbb{R}^{r\times n}$ and $\lambda \in \mathbb{R}$ is feasible:*

$$P_1 - P_0 > 0, \; P_0 > 0, \; \lambda > 0, \quad (33)$$

$$\left[\begin{array}{ccc|c|c|c} & & & P_1 B_w & M_B B_v & L^T \\ \hline & & & P_0 B_w & M_B B_v & -M_L^T \\ \hline & & & 0 & 0 & 0 \\ \hline B_w^T P_1 & B_w^T P_0 & 0 & -\gamma^2 I_{n_w} & 0 & 0 \\ \hline B_v^T M_B^T & B_v^T M_B^T & 0 & 0 & -\gamma^2 I_{n_v} & 0 \\ \hline L & -M_L & 0 & 0 & 0 & -I_r \end{array}\right] \leq 0 \quad (34)$$

In equation (34), the $(1:3, 1:3)$ block is the same as in equation (15).

With the solution P_0, P_1, M_A, M_B and M_L found, the matrices of the filter are given as before. Moreover, the L_2 induced norm satisfies $\rho \leq \gamma$ for all admissible uncertainties.

We point out again that the conditions in theorem 13 are necessary and sufficient for finding the upper bound γ provided in lemma 12. The result for structured uncertainty can be obtained similarly, but that is only sufficient.

5 Example

In this section, an example is used to illustrate the result for dynamic systems with unstructured norm bounded uncertainty. The example has been used in [18], and is therefore useful for comparing with former results. Consider the following system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 1+5\delta(t) \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 1 & -1 \end{bmatrix} x(t) + v(t) \end{aligned}$$

where $w(t) \in \mathbb{R}^2$ and $v(t) \in \mathbb{R}$ are white Gaussian noise, with variance-covariance matrix I_2 and 1, respectively.

The numerical design was implemented using the result of theorem 5. The software used are SeDumi [22] (running in Matlab) and lmitool [6], giving the following results:

$$\begin{aligned} A_f &= \begin{bmatrix} -22.90 & 0.1294 \\ -18.68 & -0.9325 \end{bmatrix}, \; B_f = \begin{bmatrix} -0.5675 \\ -0.4831 \end{bmatrix}, \\ L_f &= \begin{bmatrix} -39.44 & -0.8102 \\ -0.8102 & -0.9930 \end{bmatrix}. \end{aligned}$$

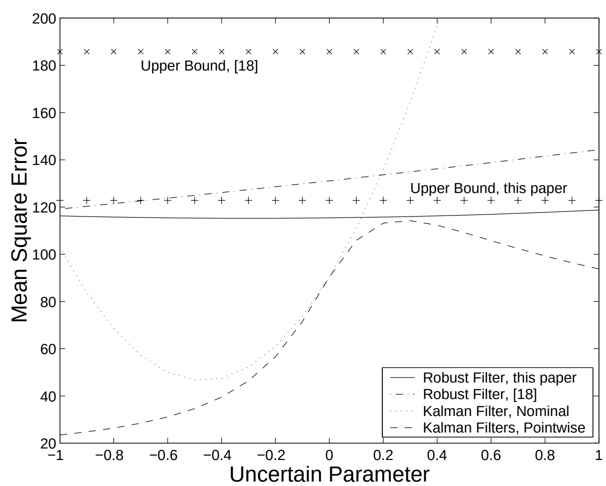

Figure 1: Comparison of Robust and Kalman Filters

Figure 1 shows the steady state mean square estimation error as a function of δ for 3 filters: the Kalman filter designed for the nominal model ($\delta = 0$), the robust H_2 filter presented in this paper and the robust H_2 filter from [18]. The upper bounds for both robust filters are also shown. The lowest curve is the optimal pointwise mean square error, obtained by designing Kalman filters at each fixed value of $\delta \in [-1, \; 1]$.

The graph shows that at $\delta = 0$, the Kalman filter results in smaller mean square error, but the robust H_2 filter designed using theorem 5 has much better worst case mean square error. This robust filter also achieves smaller mean square error than the robust filter from

[18]. The extremely small gap at $\delta = 0.25$ between the mean square error performance of our robust H_2 filter and the optimal (at $\delta = 0.25$) filter is also impressive.

6 Conclusion

In this paper, we studied the design problem of the worst case H_2 and H_∞ filter for dynamic systems with structured and unstructured, norm bounded and time varying uncertainties. The uncertain system was represented by LFTs. Based on a single quadratic Lyapunov function and a nonlinear transformation, both problems were reduced to convex optimization problems involving linear matrix inequalities (LMIs). It is also shown that for the norm bounded unstructured case, this LMI approach is less conservative, or at least as good as, the result got by Riccati equations. The example confirms the results.

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References

[1] D. P. Bertsekas and I. B. Rhodes. Recursive state estimation for a set-membership description of uncertainty. *IEEE Trans. Auto. Contr.*, AC-16(2):117–128, April 1971.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, 1994.

[3] C. E. de Souza and A. Trofino. A linear matrix inequality approach to the design of robust H_2 filters. In L. EL Ghaoui and S.-I. Niculescu, editors, *Advances in Linear Matrix Inequality Methods in Control*, pages 175–185. SIAM, December 1999.

[4] L. El Ghaoui and G. Calafiore. Robust filtering for discrete-time systems with bounded noise and parametric uncertainty. *IEEE Trans. Auto. Contr.*, 46(7):1084–1089, July 2001.

[5] L. El Ghaoui and J.P. Folcher. Multiobjective robust control of LTI systems subject to unstructured perturbations. *Sys. Contr. Letter*, 28(1):23–30, June 1996.

[6] L. El Ghaoui, R. Nikoukhah, F. Delebecque, and J.-L. Commeau. lmitool, version 2.2. [Online], 2001. Available via: http://robotics.eecs.berkeley.edu/ elghaoui/.

[7] M. Fu, C. E. de Souza, and L. Xie. H_∞ estimation for uncertain systems. *Int. J. Rob. Nonlinear Contr.*, 2:87–105, 1992.

[8] Minyue Fu, C.E. de Souza, and Zhi-Quan Luo. Finite-horizon robust kalman filter design. *IEEE Trans. Signal Proc.*, 49(9):2103–12, September 2001.

[9] J. C. Geromel. Optimal linear filtering under parameter uncertainty. *IEEE Trans. Signal Proc.*, 47:168–175, 1998.

[10] J. C. Geromel, J. Berussou, G. Garcia, and M. C. De Oliveira. H_2 and H_∞ robust filtering for discrete-time linear systems. *SIAM J. Contr. Optim.*, 38(5):1353–1368, 2000.

[11] H. Li and M. Fu. A linear matrix inequality approach to robust H_∞ filtering. *IEEE Trans. Signal Proc.*, 45(9):2338–2350, 1997.

[12] K. M. Nagpal and P. P. Khargonekar. Filtering and smoothing in an H_∞ setting. *IEEE Trans. Auto. Contr.*, 36(2):152–166, February 1991.

[13] A. Packard and J. C. Doyle. The complex structured singular value. *Automatica*, 29:71–109, 1993.

[14] R.M. Palhares and P.L.D Peres. Optimal filtering schemes for linear discrete-time systems - an LMI approach. *Int. J. Syst. Science*, 29(6):587–593, 1998.

[15] R.M. Palhares and P.L.D Peres. LMI approach to the mixed H_2/H_∞ filtering design for discrete-time uncertain systems. *IEEE Trans. Aeros. Electr. Systems*, 37(1):292–296, January 2001.

[16] I. R. Petersen and D. C. McFarlane. Optimal guaranteed cost control and filtering for uncertain linear systems. *IEEE Trans. Auto. Contr.*, 39(9):1971–1977, 1994.

[17] I. R. Petersen and D. C. McFarlane. Optimal guaranteed cost filtering for uncertain discrete time systems. *Int. J. Rob. Nonlinear Contr.*, 6(4):267–280, 1996.

[18] I. R. Petersen and A. V. Savkin. *Robust Kalman Filtering for Signals and Systems with Large Uncertainties.* Birkhauser, Boston, 1999.

[19] A. H. Sayed. A framework for state-space estimation with uncertain models. *IEEE Trans. Auto. Contr.*, 46(7):998–1013, July 2001.

[20] C. W. Scherer. Robust mixed control and linear parameter-varying control with full block scalings. In L. EL Ghaoui and S.-I. Niculescu, editors, *Advances in Linear Matrix Inequality Methods in Control*, pages 187–207. SIAM, December 1999.

[21] U. Shaked and C. E. de Souza. Robust minimum variance filtering. 43:2474–2483, 1995.

[22] Jos F. Sturm. Sedumi. [Online], 2001. available via: http://fewcal.kub.nl/sturm/software/sedumi.html.

[23] Y. Theodor and U. Shaked. Robust discrete-time minimum-variance filtering. *IEEE Trans. Signal Proc.*, 44(2):181–189, 1996.

[24] F. Wu and A. Packard. Optimal LQG performance of linear uncertain systems using state-feedback. In *Proc. Amer. Conf. Contr.*, volume 6, pages 4435–4439, June 1995.

[25] L. Xie and E. de Souza. On robust filtering for linear systems with parameter uncertainty. In *Proc. Conf. Decision Contr.*, pages 2087–92, New Orleans, LA, 1995.

[26] L. Xie, E. de Souza, and M. Fu. H_∞ estimation for discrete-time linear uncertain systems. *Int. J. Rob. Nonlinear Contr.*, 1:111–123, 1991.

[27] L. Xie, Y. C. Soh, and C. E. de Souza. Robust kalman filtering for uncertain discrete-time systems. *IEEE Trans. Auto. Contr.*, 39(6):1310–1314, June 1994.

[28] K. Y. Yang, S. R. Hall, and E. Feron. Robust H_2 control. In L. EL Ghaoui and S.-I. Niculescu, editors, *Advances in Linear Matrix Inequality Methods in Control*, pages 153–173. SIAM, December 1999.

[29] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, Upper Saddle River, New Jersey, 1996.