Optimal, Worst Case Filter Design via Convex Optimization[1]

Kunpeng Sun and Andrew Packard
Department of Mechanical Engineering
University of California, Berkeley, CA, 94720
{kpsun, pack}@jagger.me.berkeley.edu

***Abstract*— We propose a convex optimization method for optimal robust linear filter design. This is based on the observation that the design problem, which is infinite dimensional, is convex in the filter. It is shown that finite dimensional relaxations can be used to get arbitrary close to the optimal solution. The design procedure constitutes successive finite dimensional approximations, involving worst case analysis to get converging upper and lower bounds. Our approach differs from standard robust filtering techniques. Usually, these minimize a specific choice of upper bound of the objective function. The choice is usually well-motivated, but partially made for computational simplicity. The computational demands put forth in this paper are much larger.**

I. Introduction

Robust filter design problem has been studied extensively in last decade, namely, [6], [15] and references therein, and the text by I.R. Petersen and A.V. Savkin [13] is a comprehensive collection of Riccati based (H_2, H_∞ and set-membership) approaches. Most of these results are for time varying perturbations, and the filter design is characterized by first upper-bounding the performance measure, then selection of filter parameters to minimize the upper bound. There is no quantitative analysis on the conservativeness introduced by the use of upper bounds. Also, these bounds typically guarantee performance in the presence of (arbitrarily) time-varying parameter uncertainty. Hence, if the actual uncertainty is time-invariant, these design methods may be additionally conservative.

In our work we exploit that the robust filter design problem (with model uncertainty and noise) is convex in the filter as an operator. This fact seems unnoticed in the literature. The key here is that the optimization is carried out directly, rather than minimizing an upper bound of the objective function. Some ideas in this paper follow from the work of Boyd [3], Ghulchak and Rantzer [8] and Dahleh [5].

The difficulty in this robust filtering problem is that the optimization problem is infinite dimensional, in both variables and constraints. We show that finite dimensional approximations can be used to get suboptimal solutions with any degree of accuracy. A design algorithm is proposed, which constitutes successive finite dimensional approximations. In this design process, converging bounds for approximation errors are calculated for each finite dimensional problem, via worst case analysis and convex optimization. In the limit, we approach the optimal solution.

This paper is organized as follows: in section II a robust filtering problem for systems with structured, time invariant uncertainty is formulated. It is shown in section III that for any given $\epsilon > 0$, there exists a finite dimensional relaxation that results in an ϵ-suboptimal solution for the original infinite dimensional problem. In section IV, we give an algorithm to design a suboptimal filter using convex optimization. An example is given in section V, and conclusions are drawn in section VI.

The notation is standard. $l_2^n(\mathbb{Z}_+)$ denote the n-dimensional signals on non-negative integers with finite energy. $\mathcal{L}_\infty^{m\times n}$ denote the space of all complex-valued m by n matrix functions on the unit circle that are bounded, i.e., if $\hat{R} \in \mathcal{L}_\infty^{m\times n}$, then $\|\hat{R}\|_\infty = \operatorname{ess\,sup}_\theta \bar{\sigma}(\hat{R}(e^{j\theta})) < \infty$. The subspace of $\mathcal{L}_\infty^{m\times n}$ that admit analytic continuations in the unit disk is denoted by $\mathcal{H}_\infty^{m\times n}$. The space $\mathcal{RH}_\infty^{m\times n}$ is all real-rational functions of $\mathcal{H}_\infty^{m\times n}$. For $A \in \mathbb{C}^{m\times n}$, $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$ denote the maximum and minimum singular value of matrix A. In a normed space $(X, \|\cdot\|)$, for $x \in X$ and $r > 0$, define $\mathrm{B}(x,r) := \{y \in X : \|y - x\| < r\}$.

II. Problem formulation

In this paper, we consider uncertain discrete-time systems. The uncertain dynamic system is represented by linear fractional transformation (LFT), where the unknown part is separated from what is known in a feedback-like connection, as shown in Figure 1, and the possible values of the unknown elements are bounded:

$$z = M_{11}d + M_{12}p \quad (1)$$
$$y = M_{21}d + M_{22}p \quad (2)$$
$$q = M_{31}d + M_{32}p \quad (3)$$
$$p = \Delta q \quad (4)$$

where $z \in l_2^{n_z}(\mathbb{Z}_+)$ are signals to be estimated, $y \in l_2^{n_y}(\mathbb{Z}_+)$ are the measurements, $d \in l_2^{n_d}(\mathbb{Z}_+)$ are disturbances, and $p, q \in l_2^{n_p}(\mathbb{Z}_+)$ are perturbation signals. $M_{ij} \in \mathcal{RH}_\infty$, $i = 1, 2,\ j = 1, 2, 3$ are known transfer matrices with appropriate dimension. $\Delta \subset \mathbb{R}^{n_p\times n_p}$ is structured and norm bounded real uncertainty. To describe the uncertainty, define

$$\mathbf{\Delta} := \left\{ \operatorname{diag}[\delta_1 I_{r_1}, \ldots, \delta_s I_{r_s}] : \ \delta_i \in \mathbb{R}, \sum_{i=1}^{s} r_i = n_p \right\}$$

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The uncertainty satisfies $\Delta \in \mathbf{B_\Delta}$, where $\mathbf{B_\Delta} := \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \le 1\}$. Throughout the paper, assume this LFT is

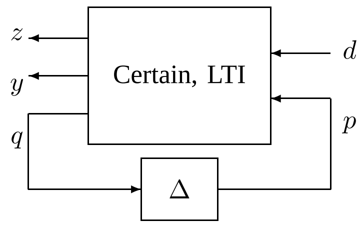

Fig. 1. Uncertain Dynamic Systems

well-posed, i.e., $(I - M_{32}\Delta)$ is invertible for all $\Delta \in \mathbf{B_\Delta}$.

The uncertain system can also be denoted in another way. Let $\hat{R}(\Delta) \in \mathcal{RH}_\infty^{n_z \times n_d}$ and $\hat{V}(\Delta) \in \mathcal{RH}_\infty^{n_y \times n_d}$ denote the uncertain transfer matrices from disturbance d to z, and from d to the output y, respectively,

$$\begin{aligned}\hat{R}(\Delta) &:= M_{11} + M_{12}\Delta(I - M_{32}\Delta)^{-1}M_{31} \\ \hat{V}(\Delta) &:= M_{21} + M_{22}\Delta(I - M_{32}\Delta)^{-1}M_{31}\end{aligned}$$

When we need to emphasize the frequency dependence, we use $\hat{R}(\cdot, e^{j\theta})$, and in some cases we use $\hat{R}$ for simplicity.

The robust $\mathcal{H}_\infty$ filtering problem is to design a filter $\hat{F} : y \to \hat{z},\ \hat{F} \in \mathcal{RH}_\infty^{n_z \times n_y}$, such that the worst case $\mathcal{H}_\infty$ norm from d to $e := z - \hat{z}$ is minimized:

$$\Lambda := \inf_{\hat{F} \in \mathcal{RH}_\infty^{n_z \times n_y}} \max_{\Delta \in \mathbf{B_\Delta}} \|\hat{R}(\Delta) - \hat{F}\hat{V}(\Delta)\|_\infty \tag{5}$$

This is illustrated in Figure 2.

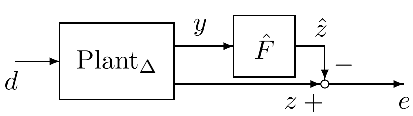

Fig. 2. Uncertain Plant and Filter

The key observation in this paper is that problem (5) is convex in the filter $\hat{F}$. Indeed, the filter appear affinely in the terms inside the norm, and the norm is a convex function, hence the objective is a convex function of the filter for any fixed uncertainty. Furthermore, the maximal function of a family of convex functions is still convex ([4]). So the robust filtering problem is a convex optimization problem.

Though the problem is convex in the filter, it is difficult to solve because the optimization problem is infinite dimensional: sets $\mathbf{B_\Delta}$ and $[0, 2\pi]$ are infinite; and set $\mathcal{RH}_\infty^{n_z \times n_y}$ is infinite dimensional. We can only find suboptimal solutions by finite dimensional approximations. Let $\mathcal{F}_\mathbf{\Delta}$ denote any finite subset of $\mathbf{B_\Delta}$. Typically, we will use M to denote the number of elements in $\mathcal{F}_\mathbf{\Delta}$, and use Δ_m (with $1 \le m \le M$) to denote a specific element. Similarly, let $\mathcal{F}_\mathbf{\Theta}$ denote any finite subset of $[0\ 2\pi]$, using N to denote the number of elements, and θ_n (with $1 \le n \le N$) to denote a specific element. Given linear independent $\{\phi_1, \phi_2, \ldots, \phi_K\} \in \mathcal{RH}_\infty^K$, define the finite dimensional subspace $\mathcal{Q}_K \subset \mathcal{RH}_\infty^{n_z \times n_y}$:

$$\mathcal{Q}_K := \{\hat{F} : \ \hat{F}(z) = \sum_{k=1}^K Q_k \phi_k(z),\ Q_k \in \mathbb{R}^{n_z \times n_y}\} \tag{6}$$

$\mathcal{Q}_K$ forms a $K \times n_z \times n_y$-dimensional subspace of $\mathcal{RH}_\infty^{n_z \times n_y}$. For $\rho > 0$, let $B_\rho \mathcal{Q}_K := \{\hat{F} \in \mathcal{Q}_K : \|\hat{F}\|_\infty < \rho\}$. The finite dimensional approximation of problem (5), parameterized by ρ, is given by:

$$\inf_{\hat{F} \in B_\rho \mathcal{Q}_K} \max_{\Delta \in \mathcal{F}_\mathbf{\Delta}} \max_{\theta \in \mathcal{F}_\mathbf{\Theta}} \bar{\sigma}\left(\hat{R}(\Delta, e^{j\theta}) - \hat{F}(e^{j\theta})\hat{V}(\Delta, e^{j\theta})\right) \tag{7}$$

In the rest of this paper, we choose basis function $\phi_k(z) = z^{k-1}$, i.e., $\hat{F}$ is a FIR filter.

To make this finite dimensional relaxation useful, we need to show that for any specified $\epsilon > 0$, can find finite sets $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$, and $K < \infty$ such that the solution is ϵ-suboptimal. On the other hand, to carry out the design, we need an iterative algorithm and nontrivial error bounds for a given finite dimensional relaxation. In the next two sections, these two points will be discussed in detail.

III. ϵ-SUBOPTIMAL FILTERS VIA FINITE DIMENSIONAL RELAXATION

In this section, we show that finite dimensional approximations (7) can be used to get suboptimal solutions of (5) for any prescribed precision ϵ. This is done by introducing the following problem:

$$\inf_{\hat{F} \in \mathcal{Q}_K} \max_{\Delta \in \mathbf{B_\Delta}} \max_{\theta \in [0, 2\pi]} \bar{\sigma}(\hat{R} - \hat{F}\hat{V}) \tag{8}$$

Let

$$\begin{aligned}\hat{F}_\mathcal{F}^* &:= \arg \min_{\hat{F} \in B_\rho \mathcal{Q}_K} \max_{\Delta \in \mathcal{F}_\mathbf{\Delta}} \max_{\theta \in \mathcal{F}_\mathbf{\Theta}} \bar{\sigma}(\hat{R} - \hat{F}\hat{V}) \\ \hat{F}_Q^* &:= \arg \min_{\hat{F} \in \mathcal{Q}_K} \max_{\Delta \in \mathbf{B_\Delta}} \max_{\theta \in [0, 2\pi]} \bar{\sigma}(\hat{R} - \hat{F}\hat{V})\end{aligned} \tag{9}$$

The argument has two steps: first, we show that for a given $\mathcal{Q}_K$, can find finite sets $\mathcal{F}_\mathbf{\Delta} \subset \mathbf{B_\Delta}$ and $\mathcal{F}_\mathbf{\Theta} \subset [0, 2\pi]$, such that $\hat{F}_\mathcal{F}^*$ is a ϵ-suboptimal solution for problem (8); second, we show there exists $K_0 < \infty$, such that when $K > K_0$, $\hat{F}_Q^*$ is a ϵ-suboptimal solution for (5).

Throughout this paper, we make two assumptions:

1) $\hat{R}(\Delta)$ and $\hat{V}(\Delta)$ are robustly stable: $\max_{\Delta \in \mathbf{B_\Delta}} \|\hat{R}(\Delta)\|_\infty < \bar{r}$, and $\max_{\Delta \in \mathbf{B_\Delta}} \|\hat{V}(\Delta)\|_\infty < \bar{v}$.
2) There exists $\underline{v} > 0$ such that $\sup_{\Delta \in \mathbf{B_\Delta}} \inf_{\theta \in [0, 2\pi]} \underline{\sigma}(\hat{V}(\Delta, e^{j\theta})) \ge \underline{v}$.

Assumption 1) is standard for most robust filter formulations. An alternative cost for unstable systems (like gap metric) would be desirable, though such results are unknown. Assumption 2) implies that $\hat{F}_Q^*$ in (9) is well defined and bounded, i.e., $\mathcal{F}_Q^* \in B_\rho \mathcal{Q}_K$:

Lemma 1: Let $\rho := \frac{2\bar{r}}{\underline{v}}$, then $\|\hat{F}_Q^*\|_\infty \le \rho$.

Proof: For any subspace $\mathcal{Q}_K \subset \mathcal{RH}_\infty^{n_z\times n_y}$, we know $0 \in \mathcal{Q}_K$, so $\hat{F} = 0$ is a feasible solution for (8). We have

$$\inf_{\hat{F}\in\mathcal{Q}_K} \max_{\Delta\in\mathbf{B_\Delta}} \|\hat{R}(\Delta) - \hat{F}\hat{V}(\Delta)\|_\infty \le \max_{\Delta\in\mathbf{B_\Delta}} \|\hat{R}(\Delta)\|_\infty \le \bar{r}$$

Then for the optimal solution $\hat{F}_Q^*$,

$$\begin{aligned}
\bar{r} &\ge \max_{\Delta} \|\hat{R}(\Delta) - \hat{F}_Q^*\hat{V}(\Delta)\|_\infty \\
&\ge \max_{\Delta}\max_{\theta} \bar{\sigma}\left(\hat{F}_Q^*(e^{j\theta})\hat{V}(\Delta, e^{j\theta})\right) - \max_{\Delta} \|\hat{R}(\Delta)\|_\infty \\
&\ge \left(\max_{\theta} \bar{\sigma}(\hat{F}_Q^*(e^{j\theta}))\right) \underline{v} - \max_{\Delta} \|\hat{R}(\Delta)\|_\infty \qquad (10)\\
&= \|\hat{F}_Q^*\|_\infty \underline{v} - \bar{r} \qquad (11)
\end{aligned}$$

Inequality (10) uses the fact that if $A \in \mathbb{C}^{n_z\times n_y}$, $B \in \mathbb{C}^{n_y\times n_d}$, then $\bar{\sigma}(AB) = \bar{\sigma}(B^TA^T) \ge \bar{\sigma}(A)\underline{\sigma}(B)$. Rearrange (11), we have $\|\hat{F}_Q^*\|_\infty \le \frac{2\bar{r}}{\underline{v}}$. ■

A. Finite uncertainty set and frequency grid

Using above result, together with the following continuity argument, we can show that finite uncertainty points and frequency grid is enough to get an ϵ-suboptimal solution.

Lemma 2: If the LFT (1) is well-posed, then the mapping $\bar{\sigma}(\hat{R}(\cdot,\cdot)) : \mathbf{B_\Delta} \times [0, 2\pi] \to \mathbb{R}$ is uniform continuous on $\mathbf{B_\Delta} \times [0, 2\pi]$, with the norm $\bar{\sigma}(\Delta) + |\theta|$. Moreover, the family of mappings $\{\bar{\sigma}(\hat{F}(\cdot)\hat{V}(\cdot,\cdot)) : \hat{F} \in B_\rho\mathcal{Q}_K\}$ are equicontinuous.

The proof of this simple lemma is given in appendix.

Theorem 3: For any $\epsilon > 0$, there exists $\delta_\Delta > 0$ and $\delta_\theta > 0$, such that for any sets $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$ satisfying $\cup_{\Delta\in\mathcal{F}_\mathbf{\Delta}} \mathrm{B}(\Delta, \delta_\Delta) \supset \mathbf{B_\Delta}$ and $\cup_{\theta\in\mathcal{F}_\mathbf{\Theta}} \mathrm{B}(\theta, \delta_\theta) \supset [0, 2\pi]$, then

$$\left| \max_{\Delta\in\mathbf{B_\Delta}} \|\hat{R} - \hat{F}_Q^*\hat{V}\|_\infty - \max_{\Delta\in\mathbf{B_\Delta}} \|\hat{R} - \hat{F}_\mathcal{F}^*\hat{V}\|_\infty \right| \le \epsilon$$

Proof: By lemma 2, $\hat{R}$ is uniform continuous, so for $\epsilon/2$, there exists δ_1, such that when $\|(\Delta_1, \theta_1) - (\Delta_2, \theta_2)\| < \delta_1$, then $\bar{\sigma}(\hat{R}(\Delta_1, e^{j\theta_1}) - \hat{R}(\Delta_2, e^{j\theta_2})) < \epsilon/2$. Similarly, by the definition of equicontinuous, for $\epsilon/2$, there exists δ_2, such that when $\|(\Delta_1, \theta_1) - (\Delta_2, \theta_2)\| < \delta_2$, then $\bar{\sigma}(\hat{F}(e^{j\theta_1})\hat{V}(\Delta_1, e^{\theta_1}) - \hat{F}(e^{j\theta_1})\hat{V}(\Delta_2, e^{j\theta_2})) < \epsilon/2$ for all $\hat{F} \in B_\rho Q_K$.

Take $\delta_\Delta = \delta_\theta = \min(\frac{\delta_1}{2}, \frac{\delta_2}{2})$. If $\mathcal{F}_\mathbf{\Delta}$ satisfies $\forall\Delta \in \mathbf{B_\Delta}$, $\exists\Delta_i \in \mathcal{F}_\mathbf{\Delta}$ that $\bar{\sigma}(\Delta_i - \Delta) < \delta_\Delta$, and $\mathcal{F}_\mathbf{\Theta}$ satisfies $\forall\theta \in [0, 2\pi]$, $\exists\theta_i \in \mathcal{F}_\mathbf{\Theta}$ that $|\theta_i - \theta| < \delta_\theta$, then $\forall\Delta,\ \theta$,

$$\begin{aligned}
&\Big| \|\hat{R}(\Delta, e^{j\theta}) - \hat{F}(e^{j\theta})\hat{V}(\Delta, e^{j\theta})\|_\infty - \\
&\qquad \|\hat{R}(\Delta_i, e^{j\theta_i}) - \hat{F}(e^{j\theta_i})\hat{V}(\Delta, e^{j\theta_i})\|_\infty \Big| \\
\le\ & \|\hat{R}(\Delta, e^{j\theta}) - \hat{R}(\Delta_i, e^{j\theta_i})\|_\infty + \\
&\qquad \|\hat{F}(e^{j\theta_i})\hat{V}(\Delta, e^{j\theta_i}) - \hat{F}(e^{j\theta})\hat{V}(\Delta, e^{j\theta})\|_\infty \\
\le\ & \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

This is true for any Δ and θ, so $\forall\hat{F} \in B_\rho\mathcal{Q}_K$,

$$\left| \max_{\Delta\in\mathbf{B_\Delta}} \max_{\theta\in[0,2\pi]} \bar{\sigma}(\hat{R} - \hat{F}\hat{V}) - \max_{\Delta\in\mathcal{F}_\mathbf{\Delta}} \max_{\theta\in\mathcal{F}_\mathbf{\Theta}} \bar{\sigma}(\hat{R} - \hat{F}\hat{V}) \right| < \epsilon$$

Particularly, for $\hat{F}_Q^*$ and $\hat{F}_\mathcal{F}^*$:

$$\max_{\Delta\in\mathcal{F}_\mathbf{\Delta}} \max_{\theta\in\mathcal{F}_\mathbf{\Theta}} \bar{\sigma}(\hat{R} - \hat{F}_Q^*\hat{V}) \le \max_{\Delta\in\mathbf{B_\Delta}} \max_{\theta\in[0,2\pi]} \bar{\sigma}(\hat{R} - \hat{F}_Q^*\hat{V}) + \epsilon$$

$$\max_{\Delta\in\mathbf{B_\Delta}} \max_{\theta\in[0,2\pi]} \bar{\sigma}(\hat{R} - \hat{F}_\mathcal{F}^*\hat{V}) \le \max_{\Delta\in\mathcal{F}_\mathbf{\Delta}} \max_{\theta\in\mathcal{F}_\mathbf{\Theta}} \bar{\sigma}(\hat{R} - \hat{F}_\mathcal{F}^*\hat{V}) + \epsilon$$

On the other hand, the following are true by the definition of $\hat{F}_Q^*$ and $\hat{F}_\mathcal{F}^*$:

$$\max_{\Delta\in\mathbf{B_\Delta}} \max_{\theta\in[0,2\pi]} \bar{\sigma}(\hat{R} - \hat{F}_Q^*\hat{V}) \le \max_{\Delta\in\mathbf{B_\Delta}} \max_{\theta\in[0,2\pi]} \bar{\sigma}(\hat{R} - \hat{F}_\mathcal{F}^*\hat{V})$$

$$\max_{\Delta\in\mathcal{F}_\mathbf{\Delta}} \max_{\theta\in\mathcal{F}_\mathbf{\Theta}} \bar{\sigma}(\hat{R} - \hat{F}_\mathcal{F}^*\hat{V}) \le \max_{\Delta\in\mathcal{F}_\mathbf{\Delta}} \max_{\theta\in\mathcal{F}_\mathbf{\Theta}} \bar{\sigma}(\hat{R} - \hat{F}_Q^*\hat{V})$$

End to end, the following inequality is true:

$$\begin{aligned}
0 \le\ & \max_{\Delta\in\mathbf{B_\Delta}} \max_{\theta\in[0,2\pi]} \bar{\sigma}(\hat{R} - \hat{F}_\mathcal{F}^*\hat{V}) - \\
& \max_{\Delta\in\mathbf{B_\Delta}} \max_{\theta\in[0,2\pi]} \bar{\sigma}(\hat{R} - \hat{F}_Q^*\hat{V}) \le 2\epsilon
\end{aligned}$$

Since ϵ is arbitrary, the proof is complete. ■

Since $\mathbf{B_\Delta}$ and $[0, 2\pi]$ are compact sets, so we can find finite $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$ based on theorem 3.

Remark 4: If $M_{32} = 0$, then for the fixed filter, the norm is convex in uncertainty Δ. Then we only need to consider the boundary of the uncertainty set. Furthermore, if the uncertainty set is a polytope at each frequency, then we only need to let $\mathcal{F}_\mathbf{\Delta}$ includes all the vertices.

B. Finite dimensional subspace of functions

In this section, we are interested in the error of uniformly approximating $\hat{F}(z) \in \mathcal{H}_\infty^{n_z\times n_y}$ by subspaces of the form $\mathcal{Q}_K$ in the $\mathcal{H}_\infty$-norm. Let $U := \{z : |z| < r\} \subset \mathbb{C}$ for $r > 1$ to denote an open disk, the approximation is very satisfactory if $\hat{F}$ is analytic on U and satisfies $\bar{\sigma}(\hat{F}(z)) < \eta\ \forall z \in U$ [16]:

Theorem 5: If there exist $r \ge 1$ and $\eta > 0$, such that $\hat{F}$ is analytic on U and satisfies $\|\hat{F}(z)\| < \eta\ \forall z \in U$, then $\sup_{\hat{F}\in B_\eta\mathcal{H}_\infty^{n_z\times n_y}} \inf_{X\in\mathcal{Q}_K} \|X - \hat{F}\|_\infty \ge \frac{\eta}{r^K}$. Moreover, the basis functions $\phi_k^*(z) := z^{k-1}$ for $k = 1, 2, \ldots, K$ are optimal, and the inequality becomes an equality.

Using this result, we can get an error bound for the robust filtering problem by using $\mathcal{Q}_K$ instead of $\mathcal{RH}_\infty^{n_z\times n_y}$:

Theorem 6: For any $\epsilon > 0$, there exists $K_0 < \infty$ such that when $K \ge K_0$, then

$$\left| \Lambda - \max_{\Delta\in\mathbf{B_\Delta}} \|\hat{R} - \hat{F}_Q^*\hat{V}\|_\infty \right| \le \epsilon \qquad (12)$$

Proof: Let $\hat{F}^0 \in \mathcal{RH}_\infty^{n_z\times n_y}$ be a $\frac{\epsilon}{2}$-suboptimal solution of problem (5). Since $\hat{F}^0 \in \mathcal{RH}_\infty^{n_z\times n_y}$, it has only finitely many poles outside the unite disk $D := \{z : |z| < 1\}$. So there exists $r > 1$, such that $\hat{F}^0$ is analytic and bounded by some sufficiently large η in rD. Take $K_0 = \frac{\ln(\eta\bar{v}) - \ln(\epsilon/2)}{\ln(r)}$, then by theorem 5, for $\hat{F}^0$ and $K \ge K_0$, there exists $\hat{F}_K \in$

$\mathcal{Q}_K$, such that $\|\hat{F}^0 - \hat{F}_K\|_\infty < \frac{\epsilon}{2\bar{v}}$. For any $\Delta \in \mathbf{B_\Delta}$, we have

$$\begin{aligned} 0 &\le \|\hat{R}(\Delta) - \hat{F}_Q^* \hat{V}(\Delta)\|_\infty - \Lambda \\ &\le \|\hat{R}(\Delta) - \hat{F}_K \hat{V}(\Delta)\|_\infty - \|\hat{R}(\Delta) - \hat{F}^0 \hat{V}(\Delta)\|_\infty + \frac{\epsilon}{2} \\ &\le \|\hat{R}(\Delta) - \hat{F}^0 \hat{V}(\Delta) - (\hat{R}(\Delta) - \hat{F}_K \hat{V}(\Delta))\|_\infty + \frac{\epsilon}{2} \\ &\le \|\big(\hat{F}^0 - \hat{F}_K\big)\hat{V}(\Delta)\|_\infty + \frac{\epsilon}{2} \\ &\le \|\hat{F}^0 - \hat{F}_K\|_\infty \bar{v} + \frac{\epsilon}{2} \le \epsilon \end{aligned}$$

since this is true for all $\Delta \in \mathbf{B_\Delta}$, the proof is complete. ■

To close this section, we combine Theorem 3 and 6, the following is immediate:

Theorem 7: For any $\epsilon > 0$, there exists $\delta_\Delta > 0$, $\delta_\theta > 0$ and $K_0 < \infty$, such that for any sets $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$ satisfying $\cup_{\Delta \in \mathcal{F}_\mathbf{\Delta}} \mathrm{B}(\Delta, \delta_\Delta) \supset \mathbf{B_\Delta}$ and $\cup_{\theta \in \mathcal{F}_\mathbf{\Theta}} \mathrm{B}(\theta, \delta_\theta) \supset [0, 2\pi]$, and $K \ge K_0$, then

$$\left| \Lambda - \max_{\Delta \in \mathbf{B_\Delta}} \|\hat{R} - \hat{F}_\mathcal{F}^* \hat{V}\|_\infty \right| \le \epsilon$$

This result also gives a criterion for choosing filter order K and sets $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$. In practice, this criterion is too conservative, and it is difficult to find K_0, δ_Δ and δ_θ. In section IV, we use a successive finite dimensional approximations to design the filter.

IV. Design algorithm and error bounds

In this section, a successive design procedure is proposed, and sets $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$ will be refined in each iteration. In this process, we also compute approximation error bounds to determine when to stop. Similar as section III, the bounds are calculated in two steps. The error bound related to the use of $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$ is calculated via worst case analysis. An overall error bound is then calculated. These bounds converge to zero in the limit.

A. An algorithm

The basic idea is to start with $\mathcal{F}_\mathbf{\Delta}$, $\mathcal{F}_\mathbf{\Theta}$ and K small. After designing the filter, check the error bounds. If they are close to zero, stop. Otherwise, add some points to $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$ or increase the filter order K, and design the filter again. The process is repeated until the desired tolerance ϵ is reached. The algorithm is as follows:

Algorithm 8:
1) Pick finite sets $\mathcal{F}_\mathbf{\Delta} \subset \mathbf{B_\Delta}$ and $\mathcal{F}_\mathbf{\Theta} \subset [0, 2\pi]$, and FIR order K;
2) Solve problem (7) with current $\mathcal{F}_\mathbf{\Delta}$, $\mathcal{F}_\mathbf{\Theta}$ and K;
3) For the result from step 2, calculate error bound Γ given in section IV-C. If $\Gamma \le \epsilon/2$, go to next step; otherwise, add some points (given in section IV-C) to $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$, and go to step 2;
4) Calculate bound Ψ given in section IV-D. If $\Psi \le \epsilon$, exit; otherwise, $K = K + 1$, and go to step 2.

If filter order K is specified, only step 1-3 are needed.

This algorithm can be thought of as a generalization of a cutting plane method, see [14], [12]. Its convergence is ensured by that of the bounds (show later), but currently, we do not have results about the convergence rate. The explicit formulation of step 1 and the calculation of bounds are addressed in the following subsections.

B. Problem in LMI form

Let $Q^K := \{Q_1, \ldots, Q_K\}$, combine (6) and (7), we have,

$$\inf_{Q^K} \max_{\Delta \in \mathcal{F}_\mathbf{\Delta}} \max_{\theta \in \mathcal{F}_\mathbf{\Theta}} \bar{\sigma}\left(\hat{R}(\Delta, e^{j\theta}) - \left(\sum_{k=1}^{K} Q_k \phi_k(e^{j\theta})\right) \hat{V}(\Delta, e^{j\theta})\right) \tag{13}$$

Problem (13) can be written in epigraph form for $t > 0$:

$$\begin{aligned} \min_{(t, Q^K)} \quad & t \\ \text{s.t.} \quad & \bar{\sigma}\left(P_{mn}(Q^K)\right) < t \quad \forall\ 1 \le m \le M,\ 1 \le n \le N \end{aligned}$$

where $P_{mn}(Q^K) := \hat{R}(\Delta_m, e^{j\theta_n}) - \left(\sum_{k=1}^{K} Q_k \phi_k(e^{j\theta_n})\right)$ $\hat{V}(\Delta_m, e^{j\theta_n})$. Recall that the maximum singular value of a matrix $H \in \mathbb{C}^{p \times q}$ can be obtained by an LMI: $\bar{\sigma}(H) < \gamma$ iff $H^* H - \gamma^2 I_q < 0$ iff $\begin{bmatrix} \gamma^2 I_q & H^* \\ H & I_p \end{bmatrix} > 0$ iff $\begin{bmatrix} \gamma^2 I_p & H \\ H^* & I_q \end{bmatrix} > 0$. By this, and using $t > 0$ instead of t^2, problem (13) is posed as a semidefinite program:

$$\begin{aligned} \min_{(t, Q^K)} \quad & t > 0 \\ \text{s.t.} \quad & G_{mn}(t, Q^K) > 0 \quad \forall\ 1 \le m \le M,\ 1 \le n \le N \end{aligned} \tag{14}$$

where $G_{mn}(t, Q^K)$ is given by

$$G_{mn}(t, Q^K) := \begin{bmatrix} t I_{n_d} & P_{mn}(Q^K)^* \\ P_{mn}(Q^K) & I_{n_z} \end{bmatrix}$$

This is a convex optimization problem involving LMIs and can be solved efficiently. The result of problem (13) is $\sqrt{t}$ obtained from (14).

C. Bounds via worst case analysis

Using worst case analysis ([1], [10]) with a fixed $\hat{F}$, we can investigate the error introduced by using $\mathcal{F}_\mathbf{\Delta}$ and $\mathcal{F}_\mathbf{\Theta}$. Recall that for any $\epsilon_0 > 0$, there exists algorithms L_{ϵ_0} and M_{ϵ_0} such that for any linear fractional G_Δ,

$$\begin{aligned} \max_{\Delta \in \mathbf{B_\Delta}} \|G_\Delta\| - \epsilon_0 < L_{\epsilon_0}(G, \mathbf{\Delta}) \le \\ \max_{\Delta \in \mathbf{B_\Delta}} \|G_\Delta\| \le M_{\epsilon_0}(G, \mathbf{\Delta}) < \max_{\Delta \in \mathbf{B_\Delta}} \|G_\Delta\| + \epsilon_0 \end{aligned}$$

The algorithm L_{ϵ_0} also yield a $\Delta^{\epsilon_0} \in \mathbf{B_\Delta}$, such that $\|G_{\Delta^{\epsilon_0}}\| = L_{\epsilon_0}(G, \mathbf{\Delta})$.

In this robust filtering problem, we know that $\hat{R}(\Delta) - \hat{F}\hat{V}(\Delta)$ is still an LFT ([11]). To make the error introduced by L_{ϵ_0} and M_{ϵ_0} relatively small, take $\epsilon_0 \leq \frac{\epsilon}{4}$. Suppose F_Q^* is the optimal solution for problem (7), we have:

$$\begin{aligned}
\alpha &:= \max_{\Delta\in\mathcal{F}_{\mathbf{\Delta}}} \max_{\theta\in\mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma}\left(\hat{R}(\Delta) - \hat{F}_Q^*\hat{V}(\Delta)\right) \\
&\leq \min_{\hat{F}\in\mathcal{Q}_K} \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \|\hat{R}(\Delta) - \hat{F}\hat{V}(\Delta)\|_\infty \\
&\leq \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \|\hat{R}(\Delta) - \hat{F}_Q^*\hat{V}(\Delta)\|_\infty =: \beta \\
&\leq M_{\epsilon_0}(\hat{R}(\Delta) - \hat{F}_Q^*\hat{V}(\Delta), \mathbf{\Delta}) =: \gamma
\end{aligned}$$

Here α and γ are computable, and their difference $\Gamma := \gamma - \alpha$ is an upper bound for error introduced by using $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$. It provides a stopping criteria for the proposed iteration generating candidate $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$ in step 2. Points added to $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$ are yielded by the algorithm L_{ϵ_0}.

$\Gamma \to 0$ as $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$ get denser, because the continuity result of lemma 2 leads to $(\beta - \alpha) \to 0$ as the process proceeds; and worst case analysis ([10]) guarantees that $(\gamma - \beta)$ is arbitrarily small.

D. A lower bound

γ obtained above is an upper bound for problem (5). If a lower bound of the same problem is obtained, then we can use their difference as a stopping criteria for algorithm 8. A lower bound η is given as follows:

$$\begin{aligned}
\eta &:= \inf_{\hat{F}\in\mathcal{RH}_\infty^{n_z\times n_y}} \max_{\Delta\in\mathcal{F}_{\mathbf{\Delta}}} \max_{\theta\in\mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma}(\hat{R} - \hat{F}\hat{V}) \qquad (15) \\
&\leq \inf_{\hat{F}\in\mathcal{RH}_\infty^{n_z\times n_y}} \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \|\hat{R} - \hat{F}\hat{V}\|_\infty \\
&\leq M_{\epsilon_0}(\hat{R}(\Delta) - \hat{F}_Q^*\hat{V}(\Delta), \mathbf{\Delta}) = \gamma
\end{aligned}$$

Problem (15) has infinite number of variables, but in a special case, it is finite dimensional, and η can be computed:

Lemma 9: If we choose basis functions $\left\{z^{k-1}\right\}_1^\infty$ for $\mathcal{RH}_\infty^{n_z\times n_y}$, and the frequency grid $\mathcal{F}_{\mathbf{\Theta}}$ is uniform, i.e., $\left\{\theta_n = \frac{2\pi(n-1)}{N}\right\}_1^N$, then problem (15) is equivalent to

$$\inf_{Q^N} \max_{\Delta\in\mathcal{F}_{\mathbf{\Delta}}} \max_{\theta\in\mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma}\left[\hat{R} - \left(\sum_{k=1}^N Q_k e^{j(k-1)\theta}\right)\hat{V}\right] \qquad (16)$$

Proof: If $\phi_k(z) = z^{k-1}$ and $\theta_n = \frac{2\pi(n-1)}{N}$, then when $k = pN+q > N$, where p, q are positive integers and $q < N$, we have

$$\begin{aligned}
&\phi_k(e^{j\theta_n})\hat{V}(\Delta_m, e^{j\theta_n}) \\
=\ & e^{j\frac{2\pi(n-1)}{N}(pN+q-1)}\hat{V}(\Delta_m, e^{j\theta_n}) \\
=\ & e^{j\frac{2n\pi(n-1)}{N}(q-1)}\hat{V}(\Delta_m, e^{j\theta_n}) \\
=\ & \phi_q(e^{j\theta_n})\hat{V}(\Delta_m, e^{j\theta_n})
\end{aligned}$$

Hence only N variables are needed, and the problem is finite dimensional. ■

Let $\Psi := \gamma - \eta$. This is a bound for the approximation error. In the particular case of lemma 9, Ψ converges to zero because $(\gamma - \alpha) \to 0$, and $(\alpha - \eta) \to 0$ as K increases.

V. EXAMPLE

The example here is an uncertain linear system in state space form with real parameter uncertainties. To transform this back to form (1–4) is an easy exercise. Software tools used to solve the optimization problem (in LMIs) are Yalmip (parser) [9] and SeDuMi (solver) [17] running in Matlab.

Consider the following system:

$$\begin{aligned}
x(k+1) &= \begin{bmatrix} 0.7 & 0.5+0.5\delta \\ -0.5 & 0.6 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(k) \\
y(k) &= \begin{bmatrix} 1 & 0.4 \end{bmatrix} x(k) + 0.2\, d(k) \\
z(k) &= x(k)
\end{aligned}$$

where $\delta \in [-1, 1]$ and $d \in l_2$, i.e., is energy bounded.

To illustrate the effects of FIR order, we specify the length of the FIR filter, and only step 1-3 of algorithm 8 are used. To give some general ideas, the result for a 2nd-order FIR filter is as follows. With stop tolerance $\Gamma \leq 0.01$, and starting with ${\mathcal{F}_{\mathbf{\Delta}}}^0 = \{0\}$, the final uncertainty set is $\mathcal{F}_{\mathbf{\Delta}} = \{0, 1, -1\}$ and the worst case performance is 2.730. The two taps are:

$$Q_1 = \begin{bmatrix} 0.6351 \\ 0.9004 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.3503 \\ -0.8653 \end{bmatrix}.$$

Table I summarizes the performance of FIR filters with different orders. It shows that the filter performs better as its order increases. It also shows that that the size of uncertainty set increases as the order of the filter increases. The last column of Table I shows the lower bound η of Λ.

TABLE I

ROBUST FIR FILTERS WITH DIFFERENT ORDER

	Upper bound γ	$\#\{\mathcal{F}_{\mathbf{\Delta}}\}$	Lower bound η
FIR2	2.730	3	1.585
FIR5	1.732	4	1.592
FIR25	1.620	7	1.604

Table II compares nominal and worst case performance of robust FIR filter in this paper with other filter design techniques: H_∞ filter [2] designed for nominal model ($\delta = 0$); robust H_∞ filter from [7] and the method proposed in [18]. Note that the worst case performance in this table are calculated directly, and they are not upper bounds. It shows that when the uncertainty is time invariant, the robust FIR filter outperform other robust filters.

TABLE II

PERFORMANCE COMPARISON OF VARIOUS FILTERS

	FIR25	H_∞ ($\delta = 0$)	[7]	[18]
Nominal	1.600	0.833	5.586	3.038
Worst case	1.618	7.884	6.701	4.105

Figure 3 compares the performance of different filters further. It shows the H_∞ norm from disturbance to estimation error as a function of δ. The lowest curve is the optimal point-wise H_∞ filter, designed at each fixed value of $\delta \in [-1,\ 1]$.

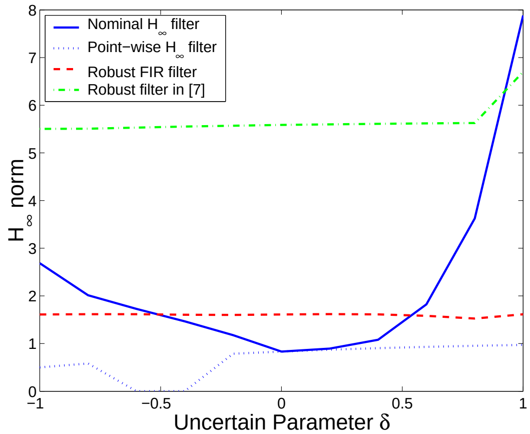

Fig. 3. Comparison of Various Filters

The graph shows that at $\delta = 0$, the H_∞ filter results in smaller norm, but the robust FIR filter proposed in this paper has much better worst case performance. This robust FIR filter also achieves better performance than the robust filter from [7] and [18].

VI. Conclusion

In this paper, we show that the robust linear filter design can be cast as a convex optimization problem. The use of finite dimensional approximations are justified and an algorithm is given to carry out the design. In the limit, the result approaches to the optimal solution. An example shows the effectiveness of the proposed algorithm. Future work will address more about the set refinement strategy, which will dramatically affect the computational complexity, as well as robust filters for systems with unmodelled dynamics, robust input design, etc..

VII. References

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Appendix

To prove lemma 2, we need to show the following lemma:

Lemma 10: The family of functions $B_\rho \mathcal{Q}_K =$

$$\left\{ \hat{F}(z) = \sum_{k=1}^{K} Q_k z^{k-1} : Q_k \in \mathbb{R}^{n_z \times n_y}, \|\hat{F}\|_\infty \leq \rho \right\}$$

are equicontinuous.

Proof: Since $\|\hat{F}\|_\infty \leq \rho$, we have

$$\begin{aligned} \sup_{\theta\in[0,2\pi]} \left(\sum_{k=1}^{K} Q_k e^{j(k-1)\theta} \right) (\cdot)^* &\leq \rho^2 I \\ \int_0^{2\pi} \left(\sum_{k=1}^{K} Q_k e^{j(k-1)\theta} \right) (\cdot)^* \, d\theta &\leq 2\pi\rho^2 I \\ \sum_{k=1}^{K} \sum_{l=1}^{K} \int_0^{2\pi} Q_k Q_l^T e^{j(k-l)\theta} \, d\theta &\leq 2\pi\rho^2 I \end{aligned}$$

Notice that

$$\int_0^{2\pi} e^{j(k-l)\theta} d\theta = \begin{cases} 2\pi & k = l \\ 0 & k \neq l \end{cases}$$

so, $\sum_{k=1}^{K} Q_k Q_k^T \leq \rho^2 I$, i.e., each Q_k must be bounded. Since $\{e^{j(k-1)\theta}, k = 1, 2, \ldots, K\}$ is equicontinuous, and Q_k are finite, so $B\mathcal{Q}_K$ is also equicontinuous. ■

Here is a proof of lemma 2.

Proof: [Proof of lemma 2] To show $\hat{R}(\Delta, e^{j\theta})$ is uniformly continuous, it is enough to show that $\hat{R}(\Delta, e^{j\theta})$ is continuous, since the set $\mathbf{B_\Delta} \times [0, 2\pi]$ is compact and $\bar{\sigma}(\cdot)$ is a continuous mapping. $\hat{R}(\Delta, e^{j\theta}) =$

$$M_{11}(e^{j\theta}) + M_{12}(e^{j\theta})\Delta(I - \Delta M_{32}(e^{j\theta}))^{-1} M_{31}(e^{j\theta})$$

Since the LFT is well posed, each individual part of $\hat{R}(\Delta, e^{j\theta})$ is continuous, so it is continuous.

Similar arguments show that $\hat{V}(\Delta, e^{j\theta})$ is uniformly continuous. Together with lemma 10, we conclude that the family of functions $\left\{ \hat{F}(e^{j\theta})\hat{V}(\Delta, e^{j\theta}) : \hat{F} \in B_\rho \mathcal{Q}_K \right\}$ is equicontinuous. ■