Results on Worst-Case Performance Assessment

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1 Worst-Case Norm

Rather than focus on computation of robust stability conditions for system with real parametric uncertainty, [3], [4], [10], [11], [12], [13], we consider worst-case performance due to real parametric uncertainty. For the purpose of this paper, we focus on a constant matrix problem. Given positive integers $k_1, k_2, \ldots, k_n$, with $k := k_1 + k_2 + \cdots + k_n$. Associated with the indices k_i, $\mathcal{D}$ denotes the operation which takes $(\delta_1, \ldots, \delta_n) =: \delta \in \mathbf{R}^n$ into the $k \times k$ diagonal matrix $\mathcal{D}(\delta) := \text{diag}\,[\delta_1 I_{k_1}, \ldots, \delta_n I_{k_n}]$. All such matrices are denoted by $\mathbf{\Delta}$, $\mathbf{\Delta} := \{\mathcal{D}(\delta) : \delta \in \mathbf{R}^n\}$, and the unit-ball is denoted $\mathbf{B_\Delta} := \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq 1\}$. Note that $\Delta \in \mathbf{B_\Delta}$ if and only if $-1 \leq \delta_i \leq 1$ for $i = 1, \ldots, n$. Given $M \in \mathbf{C}^{(k+n_e)\times(k+n_d)}$, let M_{ij} be the obvious block 2×2 partitioning with $M_{11} \in \mathbf{C}^{k\times k}$ and $M_{22} \in \mathbf{C}^{n_e \times n_d}$. For $\gamma > \bar{\sigma}(M_{22})$, define $M_{22,\gamma} := \gamma^2 I - M_{22}M_{22}^*$. For $\Delta \in \mathbf{\Delta}$, with $I - M_{22}\Delta$ invertible, define $F_u(M, \Delta) := M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12}$. Assuming $I - M_{22}\Delta$ is nonsingular for all $\Delta \in \mathbf{B_\Delta}$, solve (or approximately solve) the maximization $\max_{\Delta \in \mathbf{B_\Delta}} \bar{\sigma}[F_u(M, \Delta)]$.

2 Lower Bound

The lower bound strategy we propose consists of an exact maximization for the 1-parameter problem, and iterative coordinatewise maximization for a general many-parameter case. Suppose $M \in \mathbf{C}^{(k+n_e)\times(k+n_d)}$, partitioned as usual. Suppose that $(I - \delta M_{11})$ is nonsingular for all $-1 \leq \delta \leq 1$. Then the optimization

$$\max_{-1\leq\delta\leq1} \bar{\sigma}\left[M_{22} + M_{21}\delta\,(I - \delta M_{11})^{-1} M_{12}\right]$$

is well-posed. Mimicing Hamiltonian methods for state-space $\mathcal{H}_\infty$ norm calculation, [1], [2], yields:

Lemma 2.1 *Take M as above, and $\gamma > \bar{\sigma}(M_{22})$. If there is a $\delta \in [-1, 1]$ such that $F_u(M, \delta I_k)$ has a singular value equal to γ, then the matrix H_γ*

$$\begin{bmatrix} M_{11} & M_{12}M_{12}^* \\ 0 & M_{11}^* \end{bmatrix} + \begin{bmatrix} M_{12}M_{22}^* \\ M_{21}^* \end{bmatrix} M_{22,\gamma}^{-1} \begin{bmatrix} M_{21} & M_{22}M_{12}^* \end{bmatrix}$$

has a real eigenvalue λ satisfying $|\lambda| \geq 1$ ($\delta = \frac{1}{\lambda}$).

The real eigenvalues of the $2k \times 2k$ complex matrix H_γ give limited, yet useful information about the sublevel sets of the function $f(\delta) := \bar{\sigma}[F_u(M, \delta I_k)]$. This leads to an iterative algorithm, which tightly bounds the maximum by repeatedly computing the eigenvalues of H_γ for increasing γ. Controllability/observability assumptions on the pairs (M_{11}, M_{12}) and (M_{11}, M_{21}) render the theorem necessary and sufficient.

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An iterative procedure for $n > 1$ begins by initializing all δ_i at each midpoint, namely 0. Then, iterate on the following steps: Pick a random ordering of the integers 1 through n; Cycle through the random ordering, holding all δ's fixed at their *current* values, except the "free" one identified by the random ordering. Maximize over the free variable using the one-dimensional algorithm. As the 1-d maximization is completed, set the *current* value of the "free" δ to the maximizer.

The procedure exploits the 1-dimensional maximization, and is an ascent method, so a lower bound on the maximum is always obtained. While each successive step is not guaranteed to result in improvement, it always will if any improvement is possible in that particular coordinate direction.

We believe the main drawback is that a very explicit representation of M is required, since H_γ must be formed, and real eigenvalues computed, in contrast to the power-method, [6], [7], [15], [16], [17], where only computations of Mx and M^*v, given x and v, are needed.

2.1 Complex Uncertainties

If the problem includes complex uncertainties, it is possible to individually maximize over them as well. Full-blocks can be maximized by solving a very simple LMI, repeated-scalar blocks using state-space discrete-time $\mathcal{H}_\infty$. We have found this approach to be inferior to existing complex power methods. Modifying the complex-μ power method, [8], along the lines in [6] works better (see Section 2.2).

2.2 Initial Computational Experience

In theoretical papers on μ, algorithms are presented which parametrize all matrices having the property that μ equals the upper bound, and by simple scaling be 1. For testing purposes, one can generate these matrices, and call lower-bound routines, and assess the quality of the lower bound, at least on these types of matrices.

In each case, we computed the lower bound on the maximization for 300 matrices. This was done in an iterative fashion, holding complex blocks fixed, and maximizing (as described) cyclically over the real parameters, followed by holding the real parameters fixed, and using the complex power method to maximize over all of the complex blocks. Repeating this a few times constituted the algorithm. In the examples, the maximum is known (analytically) to be 1. The percentage of cases for which the lower bound is below a certain value is shown.

In the first figure, the number of real blocks (n) varies from 2 to 12, with each $k_i = 1$. Also, there are two scalar complex uncertainty blocks, and the performance dimensions are $n_e = n_d = 2$. In the 2nd figure, n again varies from 2 to 12, with each $k_i = 3$. As before, there are two scalar complex uncertainty blocks, and the performance dimensions are $n_e = n_d = 2$. In figure 3, n varies from 6 to 25, with each

$k_i = 4$. Here, there are no additional uncertainty blocks, and the performance dimensions are $n_e = n_d = 4$. Figure 4 shows the flop-count (averaged over the 300 runs) for the computations in figure 3. As the number of blocks increases, the effect of overhead diminishes, and we appear to get a straight-line in the loglog plot, indicating polynomial growth in the computational effort with respect to number-of-uncertainties. All of the results compare favorably to those in [6], with the exception of the flop-count, which is not discussed in [6]. It is likely that the methods in [6] involve less computations than what we present here.

3 Upper Bound

In branch-and-bound algorithms for worst-case performance, an upper bound on the worst-case performance is computed over each parameter-cube. Cubes are subdivided, and the upper bound calculation repeated. Here we show that for the well-known LMI upper bound, any feasable solution associated with a given parameter-cube is always a feasable solution for any subcube. This is probably well-known, although we were not able to find any such references.

3.1 LMI formulation

Associated with the set $\mathbf{\Delta}$ are scaling sets $\mathbf{D_\Delta}$ and $\mathbf{G_\Delta}$, defined as $\left\{\mathrm{diag}\left[D_1, \cdots, D_n\right] : 0 < D_i = D_i^* \in \mathbf{C}^{k_i \times k_i}\right\}$ and $\left\{\mathrm{diag}\left[G_1, \cdots, G_n\right] : G_i = G_i^* \in \mathbf{C}^{k_i \times k_i}\right\}$ respectively. It is well known, [5], [16], that if there exists $D \in \mathbf{D_\Delta}$ and $G \in \mathbf{G_\Delta}$ and $\beta > 0$ such that

$$M^* \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} M + \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} M - M^* \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} < \begin{bmatrix} D & 0 \\ 0 & \beta^2 I \end{bmatrix} \tag{3.1}$$

then $\max\limits_{\Delta \in \mathbf{B_\Delta}} \bar{\sigma}\left[F_u(M, \Delta)\right] < \beta$

3.2 Preliminary calculations

Suppose T and M are compatiably partitioned matrices, with the matrix product $T_{22}M_{11}$ well-defined, and square. Consider the constraints

$$\begin{bmatrix} y_1 \\ z \end{bmatrix} = T \begin{bmatrix} u_1 \\ w \end{bmatrix}, \qquad \begin{bmatrix} w \\ y_2 \end{bmatrix} = M \begin{bmatrix} z \\ u_2 \end{bmatrix} \tag{3.2}$$

Fact 3.1 *For each u_1, u_2, there exist unique vectors z, w, y_1 and y_2 solving (3.2) if and only if* $\det\left(I - T_{22}M_{11}\right) \neq 0$, *and the "star product $T{*}M$ is well-posed." $T{*}M$ is defined as the 2×2 block matrix relating the u_i to the y_i. Further, suppose that T, M and B are compatibly partitioned matrices, Assume that $T * M$ is well-posed, and that $M * B$ is well-posed. Then $(T*M)*B$ is well-posed if and only if $T*(M*B)$ is well-posed. Under these conditions, $(T*M)*B = T*(M*B)$.*

Lemma 3.2 *Suppose $T \in \mathbf{C}^{(n_1+n_1)\times(n_1+n_1)}$ and $M \in \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)}$ are compatibly partitioned matrices, with $I - T_{22}M_{11}$ invertible. Assume T_{21} is invertible. Also, let $\alpha \in \mathbf{R}$. If $M^*M + j\alpha\left(M - M^*\right) < I$ and $T^*T + j\alpha\left(T - T^*\right) \leq I$, then*

$$\left(T * M\right)^* \left(T * M\right) + j\alpha\left[\left(T * M\right) - \left(T * M\right)^*\right] < I \tag{3.3}$$

Proof: By assumption, the star product $T{*}M$ is well-posed. Let $u_i \in \mathbf{C}^{n_i}$ be arbitrary, not both 0. Let y_i and z and w be the unique solutions to the defining star-product equations (3.2), Since T_{21} is invertible, it follows that u_2 and z are not both zero. The two hypothesis in the lemma (about M and T) respectively combine with (3.2) to give

$$\begin{aligned} w^*w + y_2^*y_2 + j\alpha\left[z^*w + u_2^*y_2 - w^*z - y_2^*u_2\right] &< z^*z + u_2^*u_2 \\ y_1^*y_1 + z^*z + j\alpha\left[u_1^*y_1 + w^*z - y_1^*u_1 - z^*w\right] &\leq u_1^*u_1 + w^*w \end{aligned}$$

Adding these, and cancelling leaves $y^*y + j\alpha\left(u^*y - y^*u\right) < u^*u$, which, since u was arbitrary, and $y = (T*M)u$, implies condition (3.3) as desired.

Remark: Suppose that $\bar{\sigma}\left(T\right) \leq 1$, and $T = T^*$. Then for all $\alpha \in \mathbf{R}$, $T^*T + j\alpha\left(T - T^*\right) \leq I$, Also, if T_{21} is not invertible, then both hypothesis on M and (3.3) are changed to nonstrict inequalities.

Lemma 3.3 *Suppose $T \in \mathbf{C}^{(n_1+n_1)\times(n_1+n_1)}$ and $M \in \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)}$ are compatibly partitioned matrices, with $I - T_{22}M_{11}$ invertible. Assume T_{21} is invertible. Also, suppose $\beta > 0$, $0 < D = D^* \in \mathbf{C}^{n_1\times n_1}$ and $G = G^* \in \mathbf{C}^{n_1 \times n_1}$ are given. If*

$$M^* \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} M + \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} M - M^* \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} < \begin{bmatrix} D & 0 \\ 0 & \beta^2 I \end{bmatrix}$$

and

$$T^* \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} T + j\left(\begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} T - T^* \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}\right) \leq \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$$

then

$$(T*M)^* \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} T{*}M + \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} T{*}M - (T*M)^* \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} < \begin{bmatrix} D & 0 \\ 0 & \beta^2 I \end{bmatrix} \tag{3.4}$$

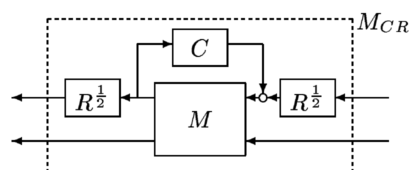
Remark: Proof is easy extension of the previous result. Suppose that $\bar{\sigma}\left(T\right) \leq 1$, $T = T^*$, and $GT_{ij} = T_{ij}G$ and $D^{1/2}T_{ij} = T_{ij}D^{1/2}$ for $1 \leq i, j \leq 2$. Then T satisfies hypothesis above. If T_{21} is not invertible, then both hypothesis on M and (3.4) are changed to nonstrict inequalities.

3.3 Feasability in cube subdivisions

For a given pair $a \in \mathbf{R}^n$ and $b \in \mathbf{R}^n$, with $a_i < b_i$ for each i, denote the cube $\mathbf{Q}_{[a,b]} := [a_1\ b_1] \times \cdots \times [a_n\ b_n] \subset \mathbf{R}^n$, and define diagonal "center" and "radius" matrices

$$C := \mathrm{diag}\left\{\frac{b_i + a_i}{2} I_{k_i}\right\}, \qquad R := \mathrm{diag}\left\{\frac{b_i - a_i}{2} I_{k_i}\right\}$$

For any M, define M_{CR} as shown



Clearly, C and R normalize the problem, giving,

$$\max_{\delta \in \mathbf{Q}_{[a,b]}} \bar{\sigma}\left[F_u\left(M, \mathcal{D}(\delta)\right)\right] = \max_{\Delta \in \mathbf{B_\Delta}} \bar{\sigma}\left[F_u(M_{CR}, \Delta)\right]$$

Suppose two cubes are given, by vectors a, b and $\tilde{a}, \tilde{b}$. Associated with each, define center and radius matrices, $C, \tilde{C}, R$ and $\tilde{R}$. Assume that $R > 0$, and define T as

$$T := \begin{bmatrix} 0 & \tilde{R}^{\frac{1}{2}} R^{-\frac{1}{2}} \\ R^{-\frac{1}{2}} \tilde{R}^{\frac{1}{2}} & R^{-\frac{1}{2}} \left(\tilde{C} - C\right) R^{-\frac{1}{2}} \end{bmatrix}$$

Note that $M_{\tilde{R}\tilde{C}} = T * M_{RC}$. The norm of T determines whether or not one cube is contained within the other. The appropriate scalar version of the result is:

Lemma 3.4 *Given $c, \tilde{c} \in \mathbf{R}$, and $r > 0, \tilde{r} \geq 0$. Then $c - r \leq \tilde{c} - \tilde{r}$ and $\tilde{c} + \tilde{r} \leq c + r$ if and only if*

$$\bar{\sigma} \begin{bmatrix} 0 & \sqrt{\frac{\tilde{r}}{r}} \\ \sqrt{\frac{\tilde{r}}{r}} & \frac{\tilde{c}-c}{r} \end{bmatrix} \leq 1$$

Hence, if $\mathbf{Q}_{[\tilde{a},\tilde{b}]} \subset \mathbf{Q}_{[a,b]}$, it follows that T satisfies equation (3.4). Combining all of these ideas yields:

Lemma 3.5 *M, R C as above, with $I - M_{11}C$ invertible. If there is $D \in \mathbf{D_\Delta}$, $G \in \mathbf{G_\Delta}$ and $\beta > 0$ such that*

$$M_{RC}^* \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} M_{RC} + \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} M_{RC} - M_{RC}^* \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} < \begin{bmatrix} D & 0 \\ 0 & \beta^2 I \end{bmatrix}$$

then it also follows that

$$M_{\tilde{R}\tilde{C}}^* \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} M_{\tilde{R}\tilde{C}} + \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} M_{\tilde{R}\tilde{C}} - M_{\tilde{R}\tilde{C}}^* \begin{bmatrix} jG & 0 \\ 0 & 0 \end{bmatrix} < \begin{bmatrix} D & 0 \\ 0 & \beta^2 I \end{bmatrix}$$

The implication is that when subdividing a cube in Divide-and-Conquer scheme, the scaling matrices obtained in the previous optimization are feasable for subdivided cube, and the optimization need not first obtain feasability.

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5 Bibilography

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