Robust Linear Filter Design via LMIs and Controller Design with Actuator Saturation via SOS Programming

by

Kunpeng Sun

B.S. (Dalian University of Technology, P.R. China) 1996
M.S. (Tsinghua University, P.R. China) 1999

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Engineering-Mechenical Engineering

in the

GRADUATE DIVISION
of the
UNIVERSITY OF CALIFORNIA, BERKELEY

Committee in charge:
Professor Andrew Packard, Chair
Professor Kameshwar Poolla
Professor Maciej Zworski

Spring 2004

The dissertation of Kunpeng Sun is approved:

Chair Date

Date

Date

University of California, Berkeley

Spring 2004

Robust Linear Filter Design via LMIs and Controller Design with Actuator Saturation via SOS Programming

Abstract

Robust Linear Filter Design via LMIs and Controller Design with Actuator Saturation via SOS Programming

by

Kunpeng Sun

Doctor of Philosophy in Engineering-Mechenical Engineering

University of California, Berkeley

Professor Andrew Packard, Chair

Robust filtering under different assumptions and formulations are considered. Robust filter design for systems described by time-varying linear fractional transformation (LFT) uncertain models is reformulated as linear matrix inequalities (LMIs) via upper bound techniques. The contribution is the treatment of norm bounded (both structured and unstructured) LFT uncertainty using LMI (rather than Riccati) methods. Furthermore, in the norm bounded unstructured uncertainty case, our results are less conservative than those by methods based on Riccati equations. Robust filter design for systems with time-invariant parameter uncertainties in a polytope is also considered, using parameter dependent Lyapunov functions to solve the problem. In both cases, we use upper bounds rather than the actual performance objectives.

We also exploit that the robust filter design problem (with model uncertainty and noise) is convex in the filter as an operator. The implication is that robust filter design can be carried out directly, rather than minimizing an upper bound of the objective function. We show that finite dimensional approximations can be used to obtain sub-optimal solutions with any degree of accuracy. A design algorithm is proposed, which is made up of successive finite dimensional approximations. This algorithm requires a worst case analysis result. A conceptual branch & bound algorithm is outlined, and a practical algorithm is given.

Polynomial state feedback controller synthesis for systems subject to actuator saturation is also considered. We are interested in two kinds of problems. The first one is to design a controller to enlarge a domain of attraction (DOA), and the second is for disturbance rejection. Sum of squares (SOS) programming is the computational tool. These synthesis problems are not convex, and ad-hoc algorithms are proposed. For linear systems with saturation, algorithms here can be used to improve available results.

Professor Andrew Packard
Dissertation Committee Chair

To my parents and Nan,

for their love and support over these years.

Contents

List of Figures

List of Tables

Acknowledgments

I would like to thank the following people for their help and support during my PhD years at Cal.

First and foremost, my deepest gratitude goes to my advisor, Professor Andy Packard, for providing a superb academic environment, for guiding my work with valuable insights and detailed comments, for working closely with me and supporting me all the time, and most importantly, for being my role model of a responsible, diligent and strongly motivated researcher.

I am also very grateful to Professor Kameshwar Poolla, for introducing the big picture of control system to me, for discussing various problems with me, and for adding fun to BCCI lab with his good sense of humor.

I am very thankful to Professor Maciej Zworski, for helping me build a solid mathematical foundation, and for his time and suggestions on my thesis.

Thanks to all my friends at BCCI lab, for exchanging research ideas, for providing computer support, and for creating a pleasant working environment. Their friendship and support have made my Cal experience very enjoyable.

Finally, I would like to thank my father, Yechen Sun, my mother, Xiaoming Ma, and my sister, Peng Sun, for their love and support all the time. Special thanks goes to my wife, Nan Yang, for loving me and supporting my work over all these years, in spite of the 2600 miles from Cal to her school, Columbia University.

Chapter 1

Introduction

Estimation problems are fundamental to control theory and signal processing, and optimal filtering techniques have been developed in last decades, see [1, 29, 46] and references therein. The main focus of this dissertation is on signal estimation for uncertain linear (time-invariant or time-varying) dynamic systems, which is also called model based robust filtering. With different system descriptions and design objectives, this problem has different formulations. In this dissertation, we show that many of these formulations can be posed as, or transformed to, convex optimization problems [1] involving linear matrix inequalities (LMIs). This particular type of optimization problems is called semidefinite programs (SDP), which can be solve very efficiently [54], and is widely used to solve other control problems. Recently, another type of convex optimization problems, sum of squares (SOS) programming, has been used to

[1]Filtering problems is a special type of control problem. But to our knowledge, there is no result on posing robust output feedback control problems as convex optimization problems.

tackle more general control problems. This is another focus of this dissertation, and we will show some results on state feedback controller design with bounded control effort on a given region (of the states), for dynamic systems with polynomial vector fields. This is important because the output of actuators are bounded in reality.

In this chapter, we introduce some basic issues in formulating robust filter design problems. Since later chapters are on various specific cases, we will give more detailed references in each individual case.

1.1 Filtering problems and robustness

The term "filtering" has been used to denote many different signal processing methods. It can be based directly on the data to be processed, picking out some specific frequency content of the data, such as low-pass filters and band-pass filters. In this thesis, we will consider model-based filters. The model-based methodology uses dynamic models to describe the relationships between signals, and the filter is then designed to achieve specific goals based on these models.

A general picture of model-based filters is shown in Figure 1.1. Here the model

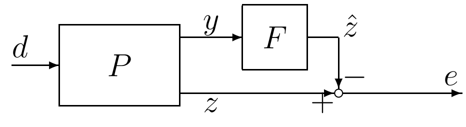

Figure 1.1: Model Based Filters

of the plant is P, d is unknown extraneous disturbance, signal z is to be estimated, and y is the measurement. The problem is to design a filter F, in general another dynamic system, taking y as its input, and the output of F is denoted as $\hat{z}$. The goal is to compute F such that $\hat{z}$ is "close" to z, i.e., estimation error e is "small". All these need more specific definitions, which we defer to later chapters. For F to be a filter, rather than a predictor or smoother, time has to be taken into account. We use t and k to denote continuous-time case and discrete-time case, respectively. Notice that when estimating $z(T)$ or $z(K)$, the signal available is up to time T or K, i.e., we can use all $y(t)$, $t \in [0, T]$ or $y(k)$, $k = 1, \ldots, K$, respectively. Otherwise, the problem is called prediction or smoothing.

A problem with model-based filtering design is that the mismatch of model and plant, which is inevitable, will cause performance degradations of the resulting filter when it is applied to the real plant. Uncertainties in the plant model may have several origins, but in general, there are two classes:

- Parametric uncertainty. In this case, the structure of the model is known, but some of the parameters are uncertain. This can be caused by parameter measurement error or identification error; linearizing a non-linear system around an operating point; or small changes of the properties of system components; etc..

- Unmodelled dynamics. This is often resulted from neglecting the dynamics at high frequency, or the structure of high order dynamics in the system is unknown.

Robust filtering techniques are attempts to relieve these situations. In some cases, improving the performance robustness means that the average performance of the design is improved. Then a probabilistic approach to describe the modelling error and solving the problem is needed. In this thesis, we will consider guaranteed performance filter design. We suppose the actual model of the plant (possibly time-varying) is not known exactly, but lies in a candidate set, and the goal is to calculate F to minimize the worst case performance over this set of possible plant models.

1.2 Problem formulation issues and problem solving tools

Uncertain model description is very important in formulating the problem, and this also affects problem solving techniques greatly. The following uncertainty models are usually used in robust filtering (and control) problems:

- Norm bounded parametric uncertainty. This is used to model parametric uncertainty, where the uncertain parameters are assumed to be in a norm bounded set, which can be either unstructured or structured.

- Polytopic parameter uncertainty. The uncertain parameters of the model lie in a polytope. This parametric uncertainty set is structured.

- Linear fractional transformations (LFT). This kind of model is more general, and can be used to model both parametric uncertainty and unmodelled dynamics. In this model, the uncertain part and the known part of plant dynamics are in a feedback-like connection.

- Integrated quadratic constraint (IQC). This model can also be used to describe both types of uncertainties.

These models can be further assumed to be either time-varying or time-invariant, since in reality, some parameter uncertainties, such as linearizing a non-linear system around an operating point, are time varying, and in some cases are time-invariant, such as parameter measurement errors.

Another factor in problem formulation is the performance measure of resulting filters. The following objectives are often used:

$\mathbf{H_2}$ filtering problem: Similar to the Kalman filter, we assume d is zero mean white noise. The H_2 performance objective is the worst case mean square error. Since the performance measure is the same as Kalman filter, this is often called Robust Kalman filter.

$\mathbf{H_\infty}$ filtering problem: In this problem, the design objective is to minimize the the worst case induced L_2 (continuous-time systems) or l_2 (discrete-time systems) operator norm from noise d to estimation error e.

Set membership estimation: The goad is to find a filter to obtain a "minimum" ellipsoid at current time, or in the steady state, that contains all possible signals (to be estimated) that are consistent with the assumed uncertainty.

Some of these objectives can be either infinite horizon or finite horizon. In the former case, one cares about steady state error, and results in a filter/controller with fixed parameters. Finite horizon design methods (only for discrete-time systems) obtain filters with time-varying parameters, such as [15]. In this thesis, we will only consider infinite horizon problems.

Full order linear filter is a frequently used filter structure. There are also cases that one considers finite-impulse-response (FIR) filters, or even try to design the optimal filter with linear structure. Different techniques are used to carry out the design, for example,

- approaches based on Riccati equations;
- approaches based on constructing Lyapunov functions, and the problem is then formulated as an SDP, also called linear matrix inequalities (LMIs);
- direct optimization, over the possible filters with given structures.

The first two type of approaches are based on finding an upper bound for the true objectives first, and then minimize this upper bound. We call them indirect methods, and call the third one direct method.

We can see that there are many different choices in formulating the problem and solving the problem. Since this research topic has appeared for more than 10 years, numerous papers have appeared. But there is still room for improvements. In this thesis, we will give some of them. These improvements are independent of each other, in that they are based on different problem formulations and solving techniques. Since the differences of references in the literature are very subtle, we will give more detailed references in each case considered in this thesis, given that they are closely related to that case.

1.3 From SDP to SOS

All the robust filtering problems in this thesis are formulated as semidefinite programs (SDP), which are very powerful, widely used tools in linear system analysis and synthesis (often used to construct quadratic Lyapunov functions). Recently, some semi-algebraic problems are formulated in a convex optimization framework, called sum of squares programming (SOS) [39], which can be used to deal with dynamic systems with polynomial vector fields and to construct polynomial Lyapunov functions. Some attempts has been used for controller synthesis [25].

The last part of this thesis, we use SOS programing as a tool to tackle state-feedback controller synthesis problem with input constrains. The goal is twofold: for linear systems with saturation, proposed algorithms can be used to improve available results, e.g., those in [24]; another goal is to see how well the SOS programming works in controller synthesis, since for linear system with saturation, we have existing methods to compare to.

1.4 Outline and contributions

As mentioned before, though most chapters of this thesis are on robust filtering problems, they are are independent in the sense that model descriptions, notations, and solving techniques are different. The outline of this thesis is as follows:

1. In Chapter 2, robust H_2 and H_∞ filter design problems for systems described by time-varying LFT uncertain models are considered. The uncertainties can be either unstructured or structured, and they are norm bounded. The main result is that after upper-bounding the objectives, the problems of minimizing the upper bounds are converted to finite dimensional convex optimization problems involving linear matrix inequalities (LMIs), and they can be solved very efficiently.

2. In Chapter 3, we consider robust H_∞ filter design for systems with time-invariant parameter uncertainties in a polytope. Systems considered in this chapter are discrete-time systems. The problem is also transformed to LMIs based on another upper bound.

3. We propose a direct method for optimal robust linear filter design in Chapter 4. This is based on the observation that the design problem, which is infinite

dimensional, is convex in the filter. It is shown that finite dimensional relaxations can be used to get arbitrary close to the optimal solution. The design procedure consists of successive finite dimensional approximations, involving worst case analysis to get upper and lower bounds. This approach differs from standard robust filtering techniques. Usually, these minimize a specific choice of upper bound of the objective function. The choice is usually well-motivated, but partially made for computational simplicity. The computational demands put forth here are much larger.

4. In Chapter 5, polynomial state feedback controller synthesis for polynomial systems with input saturations are considered. We are interested in two kinds of problems. The first one is to design a controller to enlarge a domain of attraction (DOA), and the second is about disturbance rejection. Sum of squares (SOS) programming is the computational tool. These synthesis problems are not convex, and ad-hoc algorithms are proposed.

The main contributions of this thesis are:

- The treatment of norm bounded (both structured and unstructured) LFT uncertainty using LMI (rather than Riccati) methods. In the norm bounded unstructured uncertainty case, we establish necessary and sufficient LMI conditions for finding the upper bounds, which are less conservative than those methods based on Riccati equations [42, 56]. (To our knowledge, the robust filtering problem for dynamic systems with structured LFT uncertainty has not been tackled.) These are actually extensions of well known results for systems with polytopic uncertainty.

- The robust H_∞ filter is designed for discrete-time systems with time-invariant, polytopic uncertainties. This is a simple extension of the results presented in [19], where robust H_2 problem is considered.

- A conceptual branch & bound algorithm for worst case analysis is outlined, and a practical algorithm is also given. These algorithms extend ideas in [33] to matrices with additional two full block complex perturbations (unmodelled dynamics). Another contribution here is that we propose a method to perform this calculation cross frequency.

- We exploit that the robust filter design problem (with model uncertainty and noise) is convex in the filter as an operator. This fact seems largely unnoticed in the literature. Its implication is that the optimization can be carried out directly, rather than minimizing an upper bound of the objective function. We show that finite dimensional approximations can be used to obtain suboptimal solutions with any degree of accuracy. A design algorithm of successive finite dimensional approximations is proposed.

- We extend the result of [24], with the development of a computational framework called sum of squares (SOS) programming [39], to systems with polynomial vector fields and polynomial controllers. The optimization problems formulated are not convex, and heuristic algorithms are proposed. For linear systems with saturation, algorithms here can be used to improve available results, e.g., those in [24]. This also shows that SOS programming works in controller synthesis.

Chapter 2

Robust filters for time-varying uncertain LFT systems – indirect method

Numerous papers have appeared on robust filtering problem for linear systems with time-varying parametric uncertainties. For example, the text by I.R. Petersen and A.V. Savkin [42] is a comprehensive collection of Riccati based (H_2, H_∞ and set-membership) approaches.

Robust H_2 filtering problems are often formulated in a Kalman filtering-like stochastic context, where uncertain dynamic systems are subjected to white noise. The objective of this design problem is to find filter parameters such that the worst case mean square estimation error is minimized. To our knowledge, this is done by first over-bounding the objective, and then developing techniques to minimize this bound. Petersen and McFarlane [40, 41], Xie et. al [58], Theodor and Shaked [52, 49] consider this problem with parameter uncertainties entering the system affinely and lying in an unstructured, norm bounded set. A Riccati equation based approach is used to minimize the upper bound. de Souza [9], Geromel [18] and Palhares and Peres [36] consider a polytopic uncertainty model. They convert the problem of minimizing the upper bound to a convex optimization problem involving LMIs. Fu et al. consider the finite horizon robust H_2 filtering problem in [15]. They impose a linear structure on the filter and obtain a time varying filter.

In robust H_∞ filtering problem, the filter is designed such that the worst-case induced L_2 gain from process noise to estimation error is minimized. Similar to the H_2 filtering problem, an upper bound is derived first, and then the bound is minimized using techniques based on Riccati equations or LMIs. [56, 57] and [16] consider the robust H_∞ filtering problem for norm bounded, unstructured uncertain dynamic systems using a Riccati equation approach. [17, 18] and [37] use an LMI approach to tackle the uncertain dynamic systems with polytopic uncertainties. [27] considers the H_∞ filtering problem for systems with IQC constraints, and formulates the problem with matrix inequalities, which in general, are not convex.

In this chapter, we consider robust H_2 and H_∞ filtering problems for systems with time varying, norm bounded LFT uncertainty (unstructured or structured). Both design objectives can be upper bounded, using a single quadratic Lyapunov function. The problems are then reformulated as minimizing these upper bounds. By a nonlinear transformation of variables [9, 18], both problems are converted to finite dimensional convex optimization problems involving LMIs.

The remainder of this chapter is organized as follows. In Section 2.1, the problem description is presented, and some preliminary results are given in Section 2.2. In Section 2.3, we reduce the robust H_2 filtering problem to finite dimensional optimization involving LMIs, and then further reduce to one with fewer design variables. The H_∞ filtering problem is considered in Section 2.4. An example is given in Section 2.5, and conclusions are drawn in Section 2.6.

The notation is fairly standard. $\mathbb{R}^{m\times n}$ is the set of real $m \times n$ matrices. For $M \in \mathbb{R}^{n\times n}$, $M \succ 0$ ($M \succeq 0$) indicates that $M(= M^T)$ is positive definite (semi-definite), and $M \prec 0$ means that it is negative definite. We use $\mathcal{S}_+^n$ to denote the set of $n \times n$ positive definite matrices. $tr(\cdot)$ stands for the matrix trace. $\mathcal{E}\{\cdot\}$ denotes the expectation operator. For $U \in \mathbb{R}^{n\times p}$, with $p \leq n$ and $rank(U) = p$, $U_\perp$ denotes any orthogonal complement of U, satisfying $U_\perp^T U = 0$, and $rank([U_\perp \ U]) = n$.

2.1 Problem Setup

Consider the following class of uncertain continuous-time systems

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = M(\Delta(t)) \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} \tag{2.1}$$

where $x(0) = x_0$, and $M(\Delta(t))$ is given by

$$M(\Delta(t)) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \Delta(t)(I - H\Delta(t))^{-1} \begin{bmatrix} R_1 & 0 \end{bmatrix} \quad (2.2)$$

where $x(t) \in \mathbb{R}^n$ are the states, $d(t) \in \mathbb{R}^{n_d}$ is the process and measurement noise, and $y(t) \in \mathbb{R}^m$ are the measurements. A, C, B, D, L_1, L_2, R_1 and H are known constant matrices with appropriate dimensions. The uncertainty matrix $\Delta(\cdot)$ is norm bounded, time-varying and structured or unstructured, depending on problem. We use $\mathbf{\Delta_u} := \{\Delta \in \mathbb{R}^{n_p \times n_q} : \|\Delta\| \leq 1\}$ to denote unstructured uncertainties. Structured uncertainties are denoted by $\mathbf{\Delta_s} := \{\Delta = \text{diag}(\delta_1 I_{q_1}, \ldots, \delta_l I_{q_l}, \Delta_{l+1}, \ldots, \Delta_{l+f}) : \|\Delta\| \leq 1,\ \delta_i \in \mathbb{R},\ \Delta_i \in \mathbb{R}^{q_i \times q_i} \text{and } \sum_{i=1}^{l+f} q_i = n_p = n_q\}$. For convenience, we use $\mathbf{\Delta}$ to denote both cases. L_1, L_2, R_1 and H are known constant matrices with appropriate dimensions, which specify how the elements of the nominal matrices A and C are affected by the uncertain parameter $\Delta(t) \in \mathbf{\Delta}$.

This linear fractional transformation (LFT) representation of uncertainty has great generality and is widely used in robust control theory, for instance [11, 62]. This framework includes the case when parameters perturb each coefficient of the data matrices in a polynomial or rational manner ([11]). In this paper, we assume the representation (2.1) is well-posed over $\mathbf{\Delta}$, meaning that $\det(I - H\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}$. Under this assumption, the uncertain part can be isolated from known part and the system written as,

$$\dot{x}(t) = Ax(t) + Bd(t) + L_1 p(t)$$

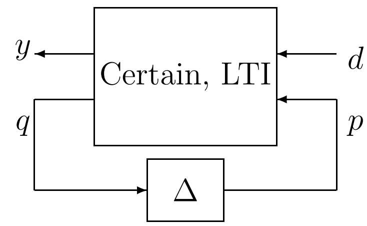

Figure 2.1: Uncertain LFT systems

$$
\begin{aligned}
y(t) &= Cx(t) + Dd(t) + L_2 p(t) \\
q(t) &= R_1 x(t) + Hp(t) \\
p(t) &= \Delta(t) q(t), \qquad \Delta(t) \in \mathbf{\Delta}
\end{aligned}
$$

where $p(t) \in \mathbb{R}^{n_p}$ and $q(t) \in \mathbb{R}^{n_q}$ are perturbation signals. This is shown in Figure 2.1.

Given $L \in \mathbb{R}^{r \times n}$, the objective is to design a linear, full order filter to estimate $z(t) := Lx(t)$. The filter structure is constrained to:

$$
\dot{\hat{x}}(t) = A_f \hat{x}(t) + B_f y(t) \tag{2.3}
$$

$$
\hat{z}(t) = L_f \hat{x}(t) \tag{2.4}
$$

where $A_f \in \mathbb{R}^{n \times n}$, $B_f \in \mathbb{R}^{n \times m}$ and $L_f \in \mathbb{R}^{r \times n}$ are constant matrices. Define estimation error $e(t) := z(t) - \hat{z}(t)$. Let $\eta(t) := [x(t)^T \ \hat{x}(t)^T]^T$ denote the states of the augmented system,

$$
\dot{\eta}(t) = [\bar{A} + \bar{L}\Delta(t)(I - H\Delta(t))^{-1}\bar{E}]\eta(t) + \bar{B}d(t) \tag{2.5}
$$

$$
e(t) = \bar{C}\eta(t) \tag{2.6}
$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix}, \ \bar{L} = \begin{bmatrix} L_1 \\ B_f L_2 \end{bmatrix}, \ \bar{E} = \begin{bmatrix} R_1 & 0 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B \\ B_f D \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} L & -L_f \end{bmatrix}.$$

In this paper, two problems are considered.

$\mathbf{H_2}$ filtering problem: We take a stochastic interpretation. In equation (2.1), assume d is zero mean white noise, with $\mathcal{E}[d(t)d(l)^T] = \delta(t-l)I_{n_d}$, where $\delta(t)$ is the Dirac Delta function. The H_2 performance objective is $\sigma := \lim_{T\to\infty} \sigma_T$, where $\sigma_T = \sup_{\Delta(\cdot)\in\mathbf{\Delta}} \mathcal{E}\left\{\frac{1}{T}\int_0^T e^T(t)e(t)\,dt\right\}$. The notation $\sup_{\Delta(\cdot)\in\mathbf{\Delta}}$ denotes the supremum over all piecewise continuous functions $\Delta : \mathbb{R} \to \mathbf{\Delta}$. The design objective is to minimize σ:

$$\min_{A_f, B_f, L_f} \quad \sigma \tag{2.7}$$

$$\text{Subject to} \quad (2.5) \text{ and } (2.6) \tag{2.8}$$

$\mathbf{H_\infty}$ filtering problem: Define the worst case induced L_2 operator norm as $\rho := \sup_{\Delta(\cdot)\in\mathbf{\Delta}} \sup_{\|d\|_2\neq 0} \frac{\|e\|_2}{\|d\|_2}$. The design objective is to minimize ρ:

$$\min_{A_f, B_f, L_f} \quad \rho \tag{2.9}$$

$$\text{Subject to} \quad (2.5) \text{ and } (2.6) \tag{2.10}$$

2.2 Preliminaries: analysis of uncertain LFT systems

To solve the problem, we need some lemmas. The first three are linear algebra, and the remaining ones pertain to uncertain linear systems.

Lemma 1 (Boyd, [6]) *Let $G \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times p}$, $V \in \mathbb{R}^{n\times q}$ and p, $q < n$. There exists a matrix $X \in \mathbb{R}^{p\times q}$ such that $G+UXV^T+VX^TU^T \succ 0$ if and only if $U_\perp^T G U_\perp \succ 0$ and $V_\perp^T G V_\perp \succ 0$.*

Notice that in Lemma 2, the result is a *necessary and sufficient* condition. We also give an alternative proof of this lemma.

Lemma 2 (El Ghaoui, [12]) *Let $T_1 = T_1^T$, T_2, T_3, T_4 be real matrices of appropriate size. We have* $\det(I - T_4\Delta) \neq 0$ *and*

$$T_1 + T_2\Delta(I - T_4\Delta)^{-1}T_3 + T_3^T(I - T_4\Delta)^{-T}\Delta^T T_2^T \prec 0 \tag{2.11}$$

for every $\Delta \in \mathbf{\Delta_u}$, if and only if there exist a scalar $\lambda > 0$ such that

$$\begin{bmatrix} T_1 + \lambda T_3^T T_3 & T_2 + \lambda T_3^T T_4 \\ T_2^T + \lambda T_4^T T_3 & \lambda(T_4^T T_4 - I) \end{bmatrix} \prec 0$$

Proof. Consider the following matrix problem:

$$\begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} T_1/2 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} x \\ q \end{pmatrix} \tag{2.12}$$

$$q = \Delta p, \qquad \|\Delta\| < 1 \tag{2.13}$$

It's easy to check that

$$y^T x + x^T y < 0, \ \ \forall x, \ q \tag{2.14}$$

is equivalent to condition (2.11). Since $y = [T_1/2 \ \ T_2][x^T \ \ q^T]^T$ and $x = [I \ 0][x^T \ \ q^T]^T$, condition (2.14) is true if and only if

$$\begin{bmatrix} T_1/2 \\ T_2^T \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} T_1/2 & T_2 \end{bmatrix} \prec 0$$

i.e.

$$\begin{bmatrix} T_1 & T_2 \\ T_2^T & 0 \end{bmatrix} \prec 0.$$

At the same time, by linear algebra, it's easy to know that condition (2.13) is equivalent to

$$p^T p > q^T q.$$

Since $p = [T_3 \ T_4][x^T \ \ q^T]^T$, above inequality can be written as:

$$\begin{bmatrix} T_3^T \\ T_4^T \end{bmatrix} \begin{bmatrix} T_3 & T_4 \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} \succ 0$$

By $\mathcal{S}$-procedure [6], we know that if (2.11) is true when (2.13) is true if and only if there exists $\lambda > 0$ such that

$$\begin{bmatrix} T_1 & T_2 \\ T_2^T & 0 \end{bmatrix} + \lambda \begin{bmatrix} T_3^T T_3 & T_3^T T_4 \\ T_4^T T_3 & T_4^T T_4 - I \end{bmatrix} \prec 0$$

This completes the proof. ■

The next lemma is the result for structured uncertainty, $\mathbf{\Delta_s}$. It is a simple generalization of the sufficient direction of Lemma 2, and we omit the $\mathcal{S}$-procedure based proof here. Associated with $\mathbf{\Delta_s}$, define structured subspaces $\mathbf{S}$ and $\mathbf{G}$ as $\mathbf{S} := \{\mathrm{diag}(S_1, \ldots, S_l, \mu_1 I_{q_{l+1}}, \ldots, \mu_s I_{q_{l+f}}) : \; S_i = S_i^T \in \mathbb{R}^{q_i \times q_i}, \; i = 1, \ldots, l\}$, and $\mathbf{G} := \{\mathrm{diag}(G_1, \ldots, G_l, 0_{q_{l+1}}, \ldots, 0_{q_{l+f}}) : \; G_i = -G_i^T \in \mathbb{R}^{q_i \times q_i}, \; i = 1, \ldots, l\}$.

Lemma 3 *Let $T_1 = T_1^T$, T_2, T_3, T_4 be real matrices of appropriate size. We have $\det(I - T_4\Delta) \neq 0$ and $T_1 + T_2\Delta(I - T_4\Delta)^{-1}T_3 + T_3^T(I - T_4\Delta)^{-T}\Delta^T T_2^T \prec 0$ for every $\Delta \in \mathbf{\Delta_s}$, if there exist block-diagonal matrices $S \in \mathbf{S}$ and $G \in \mathbf{G}$ such that $S \succ 0$ and*

$$\begin{bmatrix} T_1 + T_3^T S T_3 & T_2 + T_3^T S T_4 + T_3^T G \\ T_2^T + T_4^T S T_3 - G T_3 & T_4^T S T_4 + T_4^T G - G T_4 - S \end{bmatrix} \prec 0. \qquad (2.15)$$

Our performance bounds for uncertain systems are based on quadratic Lyapunov function arguments. These are well-known, and summarized here. For a compact set $\mathbf{\Gamma} \subset \mathbb{R}^s$, $\mathcal{F}_{\mathbf{\Gamma}}$ denotes the set of all piecewise continuous functions from $\mathbb{R}_+ \to \mathbf{\Gamma}$ with a finite number of discontinuities in any interval. Consider a linear, uncertain system, evolving as

$$\dot{x}(t) = A(\gamma(t))x(t) + B(\gamma(t))d(t) \qquad (2.16)$$

$$e(t) = C(\gamma(t))x(t) \qquad (2.17)$$

where $\gamma \in \mathcal{F}_{\mathbf{\Gamma}}$ and $A : \mathbb{R}^s \to \mathbb{R}^{n \times n}$, $B : \mathbb{R}^s \to \mathbb{R}^{n \times n_d}$, $C : \mathbb{R}^s \to \mathbb{R}^{n_e \times n}$ are continuous functions on $\mathbb{R}^s$.

Definition 1 ([26, 32]) *System (2.16-2.17) is said to be quadratically stable if there exists a positive definite matrix* $P \in \mathbb{R}^{n\times n}$ *such that* $\forall \gamma \in \Gamma$

$$A(\gamma)^T P + PA(\gamma) < 0.$$

The following lemma provides an upper bound for the worst case H_2 norm of an uncertain linear time-varying system.

Lemma 4 (Wu, [55]) *Define*

$$\mathcal{P} := \left\{ P \in \mathcal{S}_+^n : A(\gamma)^T P + PA(\gamma) + C(\gamma)^T C(\gamma) \prec 0,\ \forall \gamma \in \mathbf{\Gamma} \right\}.$$

Then system (2.16) is quadratically stable if and only if $\mathcal{P}$ *is nonempty. Furthermore, if* d *is white noise with zero mean and unit variance, then*

$$\lim_{T\to\infty} \sup_{\gamma \in \mathcal{F}_{\mathbf{\Gamma}}} \mathcal{E}\left\{ \frac{1}{T} \int_0^T e^T(t)e(t)dt \right\} \leq \inf_{P\in\mathcal{P}} \max_{\gamma\in\mathbf{\Gamma}} tr\Big(B(\gamma)^T PB(\gamma)\Big)$$

If system (2.16-2.17) is not uncertain, i.e., A, B, C *are constants, then the bound is an equality.*

An H_∞ norm upper bound for uncertain linear systems can also be derived based on the following Lyapunov function argument.

Lemma 5 (Boyd, [6]) *For system (2.16) and (2.17), if there exists a quadratic function* $V(x) = x^T Px$, $P > 0$ *and* $\beta \geq 0$, *such that for all* t *and all admissible* x *and* d,

$$\frac{d}{dt}V(x) + e^T e - \beta^2 d^T d \leq 0 \quad \forall \gamma \in \mathcal{F}_\gamma, \tag{2.18}$$

then the induced L_2 *gain for this system is less than* β.

We can express this result by a matrix inequality. In fact, when there is no uncertainty and the system is linear time invariant, the following lemma reduce to the well known bounded real lemma.

Lemma 6 *If there exists* $\beta \geq 0$ *and* $P \succ 0$ *such that*

$$\begin{bmatrix} A(\gamma)^T P + PA(\gamma) + C(\gamma)^T C(\gamma) & PB(\gamma) \\ B(\gamma)^T P & -\beta^2 I \end{bmatrix} \prec 0 \quad \forall \gamma \in \mathbf{\Gamma}, \tag{2.19}$$

then for any $\gamma \in \mathcal{F}_{\mathbf{\Gamma}}$*, the induced* L_2 *gain of (2.16-2.17) from* d *to* e *is less than* β*.*

Proof. By Lemma 5, we only need to show condition (2.19) is sufficient for (2.18). To this end, we multiply (2.19) by $[x^T d^T]$ and $[x^T d^T]^T$ from left and right, respectively, and get that $\forall x$, d and $\gamma \in \mathbf{\Gamma}$, we have

$$\begin{bmatrix} x^T & d^T \end{bmatrix} \begin{bmatrix} A(\gamma)^T P + PA(\gamma) + C(\gamma)^T C(\gamma) & PB(\gamma) \\ B(\gamma)^T P & -\beta^2 I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \leq 0$$

This means for all x, d and $\gamma \in \Gamma$,

$$x^T \left(A(\gamma)^T P + PA(\gamma) + C(\gamma)^T C(\gamma) \right) x + d^T B(\gamma)^T P x + \\ x^T P B(\gamma) d - \gamma^2 d^T d \leq 0 \tag{2.20}$$

By the dynamic system equations (2.5) and (2.6), we know

$$\begin{aligned} \frac{d}{dt} V(x(t)) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\ &= x(t)^T \left(A(\gamma(t))^T P + PA(\gamma(t)) \right) x(t) \\ &\quad + d(t)^T B(\gamma(t))^T P x(t) + x(t)^T P B(\gamma(t)) d(t) \end{aligned}$$

Notice that the algebraic condition (2.20) is true for all $x(t)$, $d(t)$ and $\gamma(t) \in \Gamma$, so it implies (2.18). This completes the proof. ■

Throughout this paper, we assume that system (2.1) is quadratically stable. This is a typical assumption of all work in this area.

2.3 Robust H_2 Filtering

2.3.1 Robust Filter Synthesis via LMIs

The robust H_2 filtering problem (2.7) is hard to solve directly, but we can change the objective to the upper bound provided in Lemma 4. Define $\bar{A}_\Delta := \bar{A} + \bar{L}\Delta(I - H\Delta)^{-1}\bar{E}$. The problem now is to design parameters A_f, B_f, L_f and $P \in \mathbb{R}^{2n\times 2n}$ to minimize the upper bound of the worst case performance measure σ:

$$\min_{P,A_f,B_f,L_f} \quad tr(\bar{B}^T P\bar{B}), \tag{2.21}$$

$$\text{Subject to} \quad \bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T\bar{C} \prec 0, \ \forall\Delta \in \mathbf{\Delta} \tag{2.22}$$

$$P \succ 0 \tag{2.23}$$

The assumption that system (2.1) is quadratically stable guarantees problem (2.21)-(2.23) is feasible.

The inequality constraints of problem (2.21)-(2.23) are not convex in P, A_f, B_f and L_f. In the next theorems, we use a nonlinear transformation ([9, 18]) to convert problem (2.21)-(2.23) to a convex one. In Theorem 1, we state the result for dynamic systems with unstructured uncertainty. The result for systems with structured uncertainty is given in Theorem 2.

Theorem 1 *Take* $\mathbf{\Delta} = \mathbf{\Delta}_u$ *and* $\gamma > 0$. *Then* $\exists\ P,\ A_f,\ B_f,\ L_f$ *such that (2.22), (2.23) and* $tr(\bar{B}^T P \bar{B}) < \gamma$ *if and only if the following LMI problem in* M_A, P_0, $P_1 \in \mathbb{R}^{n\times n}$, $M_B \in \mathbb{R}^{n\times m}$, $M_L \in \mathbb{R}^{r\times n}$, $N \in \mathbb{R}^{n_d\times n_d}$ *and* $\lambda \in \mathbb{R}$ *is feasible.*

$$tr(N) < \gamma,\ P_1 - P_0 \succ 0,\ P_0 \succ 0,\ \lambda > 0, \tag{2.24}$$

$$\left[\begin{array}{c|c|c|c} M_1 & M_2 & M_3 & L^T \\ \hline M_2^T & M_A + M_A^T & P_0 L_1 + M_B L_2 & -M_L^T \\ \hline M_3^T & L_1^T P_0 + L_2^T M_B^T & \lambda(H^T H - I_{n_p}) & 0 \\ \hline L & -M_L & 0 & -I_r \end{array}\right] \prec 0 \tag{2.25}$$

$$\begin{bmatrix} N & B^T P_1 + D^T M_B^T & B^T P_0 + D^T M_B^T \\ P_1 B + M_B D & P_1 & P_0 \\ P_0 B + M_B D & P_0 & P_0 \end{bmatrix} \succeq 0, \tag{2.26}$$

where in (2.25), M_1, M_2 *and* M_3 *are defined as follows:*

$$\begin{aligned} M_1 &= A^T P_1 + P_1 A + M_B C + C^T M_B^T + \lambda R_1^T R_1, \\ M_2 &= A^T P_0 + M_A + C^T M_B^T, \\ M_3 &= P_1 L_1 + M_B L_2 + \lambda R_1^T H. \end{aligned}$$

With solutions $(N,\ P_0,\ P_1,\ M_A,\ M_B,\ M_L)$ *found, the matrices of the filter are given by*

$$A_f = P_3^{-1} M_A (P_3^T)^{-1} P_2,\ B_f = P_3^{-1} M_B,\ L_f = M_L (P_3^T)^{-1} P_2, \tag{2.27}$$

where P_2 and P_3 are any $n \times n$ matrices with P_2 symmetric and $P_3 P_2^{-1} P_3^T = P_0$. Moreover, the asymptotic mean square estimation error satisfies $\sigma \leq tr(N)$ for all admissible uncertainties.

Proof. Partition P as

$$P = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix}, \tag{2.28}$$

where $P_1 \in \mathbb{R}^{n \times n}$. Note that, without loss of generality, we can assume P_3 is nonsingular. We can do this because for given $P \prec 0$, suppose P_3 is singular, we can slightly perturb it to make it nonsingular while keep $P \prec 0$ since it is strict. We introduce new variables, let $P_0 := P_3 P_2^{-1} P_3^T$, $M_A := P_3 A_f P_2^{-1} P_3^T$, $M_B := P_3 B_f$ and $M_L := L_f P_2^{-1} P_3^T$.

First, using Schur complement, $tr(\bar{B}^T P \bar{B}) < \gamma$ if and only if there exists $N \in \mathbb{R}^{n_d \times n_d}$ such that $tr(N) < \gamma$ and $\begin{bmatrix} N & \bar{B}^T P \\ P\bar{B} & P \end{bmatrix} \succeq 0$. This is equivalent to

$$\begin{bmatrix} N & B^T P_1 + D^T M_B^T & B^T P_3 + D^T B_f^T P_2 \\ P_1 B + M_B D & P_1 & P_3 \\ P_3^T B + P_2 B_f D & P_3^T & P_2 \end{bmatrix} \succeq 0 \tag{2.29}$$

Let $J_1 = \text{diag}\{I_d, I_n, P_2^{-1} P_3^T\}$, and multiply (2.29) by J_1^T and J_1 from left and right, respectively, we obtain (2.29) is true if and only if condition (2.26) is satisfied.

Second, write the constraint (2.22) explicitly,

$$[\bar{A} + \bar{L}\Delta(I - H\Delta)^{-1}\bar{E}]^T P + P[\bar{A} + \bar{L}\Delta(I - H\Delta)^{-1}\bar{E}] + \bar{C}^T \bar{C} \prec 0 \tag{2.30}$$

By Lemma 2, (2.30) is true for all $\Delta \in \mathbf{\Delta_u}$ if and only if $\exists\, \lambda > 0$, such that

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \bar{C}^T\bar{C} + \lambda \bar{E}^T\bar{E} & P\bar{L} + \lambda \bar{E}^T H \\ \bar{L}^T P + \lambda H^T \bar{E} & \lambda(H^T H - I_{n_p}) \end{bmatrix} \prec 0 \tag{2.31}$$

This can be written as

$$\left[\begin{array}{c|c|c} A^T P_1 + P_1 A + P_3 B_f C & A^T P_3 + P_3 A_f & P_1 L_1 + P_3 B_f L_2 \\ +C^T B_f^T P_3^T + \lambda R_1^T R_1 + L^T L & +C^T B_f^T P_2 - L^T L_f & +\lambda R_1^T H \\ \hline P_3^T A + A_f^T P_3^T + P_2 B_f C - L_f^T L & A_f^T P_2 + P_2 A_f + L_f^T L_f & P_3^T L_1 + P_2 B_f L_2 \\ \hline L_1^T P_1 + L_2^T B_f^T P_3^T + \lambda H^T R_1 & L_1^T P_3 + L_2^T B_f^T P_2 & \lambda(H^T H - I_{n_p}) \end{array}\right] < 0 \tag{2.32}$$

Let $J = \text{diag}\{I, P_2^{-1}P_3^T, I\}$, and multiply (2.32) by J^T and J from left and right, respectively, we obtain the equivalent LMI condition:

$$\left[\begin{array}{c|c|c} M_1 + L^T L & M_2 - L^T M_L & M_3 \\ \hline M_2^T - M_L^T L & M_A + M_A^T + M_L^T M_L & P_0 L_1 + M_B L_2 \\ \hline M_3^T & L_1^T P_0 + L_2^T M_B^T & \lambda(H^T H - I_{n_p}) \end{array}\right] \prec 0 \tag{2.33}$$

This equation is not an LMI yet because of the term $M_L^T M_L$. Using Schur complement [6], we can get the equivalent condition (2.25).

Finally, by Schur complement, the constraint (2.23) is true if and only if $P_2 \succ 0$ and $P_1 - P_3 P_2 P_3^T = P_1 - P_0 \succ 0$. Since P_3 is nonsingular, $P_2 \succ 0$ if and only if $P_3 P_2 P_3^T = P_0 \succ 0$. This is condition (2.24). $\sigma \leq tr(N)$ is the conclusion of Lemma 4. Thus the proof is complete. ■

The filter matrices of (2.27) can be rewritten as

$$A_f = P_3^{-1}(M_A P_0^{-1})P_3, \qquad B_f = P_3^{-1}M_B, \qquad L_f = (M_L P_0^{-1})^{-1}P_3.$$

This implies that the matrix P_3 is a change of basis of the filter states, and has no effect on the filter's input-output property. Thus, we can take $P_3 = I_n$, and corresponding parameters of the optimal filter are: $A_f = M_A P_0^{-1},\ B_f = M_B,\ L_f = M_L P_0^{-1}$.

Corollary 1 *The design problem (2.21)-(2.23) can be transformed into the following convex problem involving LMIs:*

$$\min \qquad \gamma \tag{2.34}$$

$$\text{Subject to} \quad (2.24),\ (2.25) \text{ and } (2.26) \tag{2.35}$$

Remark 1 *When $H = 0$ in (2.2) and $L = I$, the problem formulation (2.21)-(2.23) is the same as in [40, 58, 49], etc., where a particular structure, parameterized by a scalar, is imposed on the solution. The upper bound is then minimized over the scalar. In contrast, here we are solving the optimization problem directly, with the help of lemma 2. Since the result in theorem 1 is* necessary and sufficient, *the upper bound here must be less than or equal to that obtained in [40, 58, 49], with more required off-line computation as a trade off. See Lemma 2.5 of [55] for a similar argument of LMI versus Riccati-based methods.*

Remark 2 *In [12], a robust control problem with a similar uncertainty model and performance measure is considered. The formulation includes, as a special case, the* *filtering problem considered here. Nevertheless, the synthesis conditions derived in [12] are bilinear matrix inequalities, and direct substitution of our problem data into their conditions yields BMIs. The equivalence of these non-convex BMIs to our LMI conditions (14-16) is true, but not obvious. Consequently, [10], the convexity of the robust filtering problem was not noticed by the authors of [12].*

Remark 3 *When there is no uncertainty, the robust filter designed by algorithm in Theorem 1 recovers the performance of the standard Kalman filter.*

Uncertainty with block diagonal structure arises naturally in interconnected systems [33]. Taking this structure into account results in a less conservative design. In this case, the robust H_2 filtering problem can be formulated to LMI conditions similarly by using Lemma 3. We omit the proof for simplicity.

Theorem 2 *When the uncertainties are structured, i.e., $\Delta(\cdot) \in \mathbf{\Delta_s}$, Lemma 3 is used as a sufficient condition to enforce condition (2.22), conservatively replacing (2.22) with an LMI of the form (2.15). That optimization is directly solvable as the following LMI problem in M_A, P_0, $P_1 \in \mathbb{R}^{n\times n}$, $M_B \in \mathbb{R}^{n\times m}$, $M_L \in \mathbb{R}^{r\times n}$, $N \in \mathbb{R}^{n_d\times n_d}$ and $S \in \mathbf{S}, G \in \mathbf{G}$:*

$$\min_{N,P_0,P_1,M_A,M_B,M_L,S,G} \quad tr(N), \tag{2.36}$$

$$\text{Subject to } P_1 - P_0 \succ 0,\ P_0 \succ 0,\ S \succ 0,\ (2.26) \tag{2.37}$$

$$\left[\begin{array}{c|c|c|c} M_4 & M_2 & M_5 & L^T \\ \hline M_2^T & M_A + M_A^T & P_0 L_1 + M_B L_2 & -M_L^T \\ \hline M_5^T & L_1^T P_0 + L_2^T M_B^T & H^T S H + H^T G - G H - S & 0 \\ \hline L & -M_L & 0 & -I_r \end{array}\right] \prec 0, \qquad (2.38)$$

where M_2 is given as before and

$$\begin{aligned} M_4 &= A^T P_1 + P_1 A + M_B C + C^T M_B^T + R_1^T S R_1, \\ M_5 &= P_1 L_1 + M_B L_2 + R_1^T S H + R_1^T G. \end{aligned}$$

With the minimizing solution (N, P_0, P_1, M_A, M_B, M_L) found, the matrices of the filter are given as before. Moreover, the asymptotic mean square estimation error satisfies $\sigma \leq tr(N)$ for all admissible uncertainties.

2.3.2 Elimination of Filter Parameters

The number of design variables in above optimization problems can be made smaller without introducing any conservatism. In fact, variable M_A can be eliminated from theorem 1 and 2, and the alternative formulations are still convex optimization problems involving LMIs. Here we only give the result for the unstructured uncertainty case — the formula for the structured one can be obtained similarly.

Theorem 3 *When the uncertainties are unstructured, for a given $\gamma > 0$, $tr(\bar{B}^T P \bar{B}) < \gamma$ and (2.22), (2.23) are satisfied if and only if the following LMI problem in P_0,*

$P_1 \in \mathbb{R}^{n\times n}$, $M_B \in \mathbb{R}^{n\times m}$, $M_L \in \mathbb{R}^{r\times n}$, $N \in \mathbb{R}^{n_d\times n_d}$ and $\lambda \in \mathbb{R}$ is feasible:

$$(2.24),\ (2.26) \text{ and} \tag{2.39}$$

$$\left[\begin{array}{c|c} M_1 + L^T L & M_3 \\ \hline M_3^T & \lambda(H^T H - I_{n_p}) \end{array}\right] \prec 0 \tag{2.40}$$

$$\left[\begin{array}{c|c|c} M_6 & (P_1 - P_0)L_1 + \lambda R_1^T H & M_L^T \\ \hline L_1^T(P_1 - P_0) + \lambda H^T R_1 & \lambda(H^T H - I_{n_p}) & 0 \\ \hline M_L & 0 & -I_r \end{array}\right] \prec 0 \tag{2.41}$$

where

$$M_6 := A^T(P_1 - P_0) + (P_1 - P_0)A + L^T L + M_L^T L + L^T M_L + \lambda R_1^T R_1$$

After getting N, P_0, P_1, M_B, M_L and λ, find M_A such that (2.33) holds. This is an LMI feasibility problem only in M_A, which is guaranteed to be feasible.

With solutions (N, P_0, P_1, M_A, M_B, M_L) found, the matrices of the filter are given as before. Moreover, the asymptotic mean square estimation error satisfies $\sigma \leq tr(N)$ for all admissible uncertainties.

Proof. Notice that we only eliminate M_A from inequality (2.25), all other conditions and results in Theorem 1 are intact. For convenience, we consider (2.33) instead of

(2.25). Let $U^T = [I \ I \ 0_{n_p \times n}]$, $V^T = [0 \ I \ 0_{n_p \times n}]$ and

$$R := \left[\begin{array}{c|c|c} M_1 + L^T L & A^T P_0 + C^T M_B^T - L^T M_L & M_3 \\ \hline P_0 A + M_B C - M_L^T L & M_L^T M_L & P_0 L_1 + M_B L_2 \\ \hline M_3^T & L_1^T P_0 + L_2^T M_B^T & \lambda(H^T H - I_{n_p}) \end{array}\right].$$

Then equation (2.33) can be written as

$$R + U M_A V^T + V M_A^T U^T \prec 0 \tag{2.42}$$

By Lemma 1, equation (2.42) is true if and only if

$$U_\perp^T R U_\perp < 0, \ V_\perp^T R V_\perp < 0, \tag{2.43}$$

where $U_\perp$ and $V_\perp$ are orthogonal complement of U and V, and

$$U_\perp = \begin{bmatrix} I & 0 \\ -I & 0 \\ 0 & I_{n_p} \end{bmatrix}, \ V_\perp = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I_{n_p} \end{bmatrix}. \tag{2.44}$$

Writing equation (2.43) explicitly, we have (2.40) and

$$\left[\begin{array}{c|c} M_6 + M_L^T M_L & (P_1 - P_0) L_1 + \lambda R_1^T H \\ \hline L_1^T (P_1 - P_0) + \lambda H^T R_1 & \lambda(H^T H - I_{n_p}) \end{array}\right] \prec 0, \tag{2.45}$$

Equation (2.45) is not an LMI yet because of the term $M_L^T M_L$, which is equivalent to (2.41) by Schur complement.

After solving above problem, find M_A such that (2.42) holds. By Lemma 1, this feasibility problem is guaranteed to be feasible. ■

2.4 Robust H_∞ Filtering

In this section, robust H_∞ filtering problems are considered. All the results for robust H_2 filtering have counterparts for robust H_∞ filtering. Here we only give the result for systems with unstructured uncertainty, other results can be obtained similarly.

The robust H_∞ filtering problem can be reformulated using the upper bound γ given in Lemma 6, and the goal becomes minimizing γ by choosing filter parameters, A_f, B_f, L_f, and the Lyapunov variable P, such that

$$\begin{bmatrix} \bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T\bar{C} & P\bar{B} \\ \bar{B}^T P & -\gamma^2 I \end{bmatrix} \prec 0 \quad \forall \Delta \in \mathbf{\Delta}. \tag{2.46}$$

Similar to Theorem 1, by a change of variables and the result of Lemma 2, we can convert the problem to a finite dimensional one, involving LMIs.

Theorem 4 *Take $\mathbf{\Delta} = \mathbf{\Delta}_u$ and $\gamma > 0$. Then $\exists\ P \succ 0$, A_f, B_f, L_f such that (2.46) if and only if the following LMI problem in M_A, P_0, $P_1 \in \mathbb{R}^{n\times n}$, $M_B \in \mathbb{R}^{n\times m}$, $M_L \in \mathbb{R}^{r\times n}$ and $\lambda \in \mathbb{R}$ is feasible.*

$$P_1 - P_0 \succ 0,\ P_0 \succ 0,\ \lambda > 0, \tag{2.47}$$

$$\left[\begin{array}{c|c|c|c|c} & & & P_1B + M_BD & L^T \\ \hline & & & P_0B + M_BD & -M_L^T \\ \hline & & & 0 & 0 \\ \hline B^TP_1 + D^TM_B^T & B^TP_0 + D^TM_B^T & 0 & -\gamma^2 I_{n_d} & 0 \\ \hline L & -M_L & 0 & 0 & -I_r \end{array}\right] \prec 0. \qquad (2.48)$$

In equation (2.48), the $(1:3, 1:3)$ *block is the same as in equation (2.25).*

With solutions (P_0, P_1, M_A, M_B, M_L) found, the matrices of the filter are given as before. Moreover, the L_2 induced norm satisfies $\rho \leq \gamma$ for all admissible uncertainties.

Proof. By Schur complement, equation (2.46) can be written as

$$\bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T\bar{C} + \gamma^2 P\bar{B}\bar{B}^T P \preceq 0 \qquad \forall \Delta \in \mathbf{\Delta_u}$$

By Lemma 2, above condition is satisfied if and only if there exists $\lambda > 0$ and

$$\begin{bmatrix} \bar{A}^TP + P\bar{A} + \bar{C}^T\bar{C} + \gamma^2 P\bar{B}\bar{B}^TP + \lambda\bar{E}^T\bar{E} & P\bar{L} + \lambda\bar{E}^T H \\ \bar{L}^TP + \lambda H^T\bar{E} & \lambda(H^TH - I_{n_p}) \end{bmatrix} \preceq 0.$$

Using Schur complement, we can get the equivalent condition: $\lambda > 0$ and

$$\begin{bmatrix} \bar{A}^TP + P\bar{A} + \bar{C}^T\bar{C} + \lambda\bar{E}^T\bar{E} & P\bar{L} + \lambda\bar{E}^T H & P\bar{B} \\ \bar{L}^TP + \lambda H^T\bar{E} & \lambda(H^TH - I_{n_p}) & 0 \\ \bar{B}^TP & 0 & -\gamma^2 I \end{bmatrix} \preceq 0 \qquad (2.49)$$

Partition P as in (2.28), and let $J = \mathrm{diag}\{I, P_2^{-1}P_3^T, I, I\}$. Multiplying (2.49) by J^T

and J from left and right, respectively, we know (2.49) is true if and only if

$$\left[\begin{array}{ccc|c} & & & P_1B + M_BD \\ & (1:3, 1:3) \text{ of } (2.33) & & P_0B + M_BD \\ & & & 0 \\ \hline B^TP_1 + D^TM_B^T & B^TP_0 + D^TM_B^T & 0 & -\gamma^2 I_{2n} \end{array}\right] \preceq 0$$

In above, the $(1:3, 1:3)$ block is the same as the $(1:3, 1:3)$ block of matrix in (2.33). Using Schur complement again, we have the LMI condition (2.48). Condition (2.47) follows from the same argument in the proof of Theorem 1. The L_2 gain ρ of the system less than γ follows from Lemma 6. ■

We point out again that the conditions in Theorem 4 are necessary and sufficient for finding the upper bound γ provided in Lemma 6. The result for structured uncertainty can be obtained similarly, but that is only sufficient.

2.5 Example

In this section, an example is used to illustrate the result for dynamic systems with unstructured norm bounded uncertainty. The example has been used in [42], and is therefore useful for comparing with former results. Consider the following system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 1+5\delta(t) \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \end{bmatrix} d(t) \\ y(t) &= \begin{bmatrix} 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} d(t) \end{aligned}$$

where $d \in \mathbb{R}^3$ are white Gaussian noise, with variance-covariance matrix I_3.

We use the result of Theorem 1 to design a robust H_2 filter. Software tools used are SeDuMi [50], running in Matlab, and lmitool [13], giving the following results:

$$A_f = \begin{bmatrix} -22.90 & 0.1294 \\ -18.68 & -0.9325 \end{bmatrix}, \; B_f = \begin{bmatrix} -0.5675 \\ -0.4831 \end{bmatrix},$$

$$L_f = \begin{bmatrix} -39.44 & -0.8102 \\ -0.8102 & -0.9930 \end{bmatrix}.$$

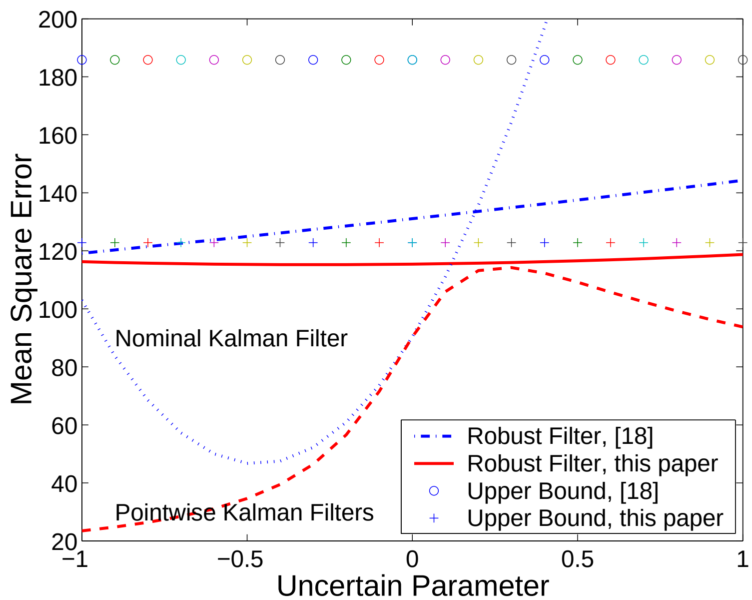

Figure 2.2: Comparison of Robust and Kalman Filters

Figure 2.2 shows the steady state mean square estimation error as a function of δ for 3 filters: the Kalman filter designed for the nominal model ($\delta = 0$), the robust H_2 filter presented in this paper and the robust H_2 filter from [42]. The upper bounds for both robust filters are also shown. These performance upper bounds hold even for time varying δ, hence are drawn as constants. The lowest curve is the optimal point-wise mean square error, obtained by designing Kalman filters at each fixed value of $\delta \in [-1,\ 1]$.

Over $-1 \leq \delta \leq 1$, the robust H_2 filter designed using Theorem 1 has the best worst case mean square error. This robust filter also achieves smaller mean square error than the robust filter from [42], at each fixed value of δ. The extremely small gap at $\delta = 0.25$ between the mean square error performance of our robust H_2 filter and the optimal (at $\delta = 0.25$) filter is also impressive.

2.6 Conclusions

In this chapter, we studied the design problem of the worst case H_2 and H_∞ filter for dynamic systems with norm bounded, time varying uncertainties. The uncertain system was represented by LFTs. Using a single quadratic Lyapunov function and a nonlinear transformation, both problems were reduced to convex optimization problems involving linear matrix inequalities (LMIs). It is also shown that for the norm bounded unstructured case, the LMI approach is less conservative than previous results.

Chapter 3

Robust filters for time-invariant uncertain systems in polytopes – indirect method

In last chapter, the uncertainties in the model are assumed to be time-varying, which will introduce some conservativeness in time-invariant uncertainty case. This problem will be considered in current chapter. Here we will consider discrete-time systems and the parameter uncertainties are assumed to be in a polytope.

For uncertain systems lying in a polytope, LMI formulations for robust filtering with a single Lyapunov function are well known, refer [18] for discrete-time systems and [9] for continuous time systems. Recently, Geromel formulated robust H_2 filtering problem as LMIs using parameter dependent Lyapunov functions, to exploit the time-invariant uncertainty. This chapter generalizes the method provided in [19] to solve H_∞ robust filter problems. This method still optimize an upper bound of the true objective.

This chapter can be looked as a transition from Chapter 2 to Chapter 4, since here we are still using an upper bound approach, and in next chapter we can design the optimal linear robust filter for time-invariant uncertainties. This chapter is similar to Chapter 2 in many aspects, so we present it tersely for simplicity.

3.1 Problem setup and preliminaries

Consider the linear time-invariant discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Bd(k) \\ y(k) &= Cx(k) + Dd(k) \\ z(k) &= Lx(k), \end{aligned}$$

where $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, $y \in \mathbb{R}^r$ and $z \in \mathbb{R}^l$. Matrices A, B, C, D and L are of appropriate dimensions. Assume that L is known and the time-invariant parameters gathered in matrix $M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ are unknown but belong to the given convex polyhedron

$$\mathcal{M} := \left\{ M(\xi) = \sum_{i=1}^{N} \xi_i M_i,\ \sum_{i=1}^{N} \xi_i = 1,\ \xi_i \geq 0 \right\}.$$

The robust H_∞ filtering problem is to design the following full order LTI filter $\mathcal{F}$:

$$\hat{x}(k+1) = A_f\hat{x}(k) + B_f y(k) \quad (3.1)$$

$$\hat{z}(k) = C_f\hat{x}(k) \quad (3.2)$$

The estimation error is $e := z - \hat{z}$. The state-space matrices for the augmented system are

$$\bar{A} = \begin{bmatrix} A & 0 \\ B_fC & A_f \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ B_fD \end{bmatrix}, \bar{C} = \begin{bmatrix} L & -C_f \end{bmatrix}, \bar{D} = 0.$$

We use $T_M(z)$ to denote the transfer function from w to estimation error e. The H_∞ filtering problem is then $\inf_{\mathcal{F}} \sup_{M\in\mathcal{M}} \|T_M\|_\infty$. The goal is to formulate this problem as LMIs by using parameter dependent Lyapunov function. We need some preliminary results to proceed.

Lemma 7 (Geromel, [18]) *Given* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, *then* $\|T_M\|_\infty^2 < \gamma$ *if and only if there exists* $P \succ 0$, *such that*

$$\begin{bmatrix} P & AP & B & 0 \\ PA^T & P & 0 & PC^T \\ B^T & 0 & I & D^T \\ 0 & CP & D & \gamma I \end{bmatrix} \succ 0.$$

The sufficient part of Lemma 7 can be extended to the case that $M \in \mathcal{M}$, which is given in the following lemma. Notice that this result obtains a Lyapunov function for every uncertain system in the polytope, which is different from results in last chapter, where one single Lyapunov function is used for all uncertainties.

Lemma 8 *For* $\gamma > 0$*,* $\|T_M\|_\infty^2 < \gamma$ *for all* $M \in \mathcal{M}$*, if there exists* $P_i \succ 0$*,* $i = 1, \dots, N$*, and* G*, such that*

$$\begin{bmatrix} P_i & A_iG & B_i & 0 \\ G^TA_i^T & G+G^T-P_i & 0 & G^TC_i^T \\ B_i^T & 0 & I & D_i^T \\ 0 & C_iG & D_i & \gamma I \end{bmatrix} \succ 0. \tag{3.3}$$

Proof. Let $P(\xi) := \sum_{i=1}^{N} \xi_i P_i$, where $\sum_{i=1}^{N} \xi_i = 1$ and $\xi_i \geq 0$. Since $P(\xi) \succ 0$, we have $(P(\xi) - G)^T P(\xi)^{-1}(P(\xi) - G) \succeq 0$, and hence $G^T P(\xi)^{-1} G \succeq G + G^T - P(\xi)$. So when (3.3) is true, we have

$$\begin{bmatrix} P(\xi) & A(\xi)G & B(\xi) & 0 \\ G^TA(\xi)^T & G^TP(\xi)^{-1}G & 0 & G^TC(\xi)^T \\ B(\xi)^T & 0 & I & D(\xi)^T \\ 0 & C(\xi)G & D(\xi) & \gamma I \end{bmatrix} \succeq$$

$$\begin{bmatrix} P(\xi) & A(\xi)G & B(\xi) & 0 \\ G^TA(\xi)^T & G+G^T-P(\xi) & 0 & G^TC(\xi)^T \\ B(\xi)^T & 0 & I & D(\xi)^T \\ 0 & C(\xi)G & D(\xi) & \gamma I \end{bmatrix} \succ 0$$

Multiplying above inequality by $T := \mathrm{diag}(I, P(\xi)G^{-T}, I, I)$ from left and T^T from right, we get

$$\begin{bmatrix} P(\xi) & A(\xi)P(\xi) & B(\xi) & 0 \\ P(\xi)A(\xi)^T & P(\xi) & 0 & P(\xi)C(\xi)^T \\ B(\xi)^T & 0 & I & D(\xi)^T \\ 0 & C(\xi)P(\xi) & D(\xi) & \gamma I \end{bmatrix} \succ 0.$$

By Lemma 7, we conclude that $\|T_M\|_\infty^2 < \gamma$ for all $M \in \mathcal{M}$. ■

3.2 Filtering result

In this section, the goal is to transform the robust H_∞ filtering problem into LMIs. To this end, some change of variables are necessary:

$$G := \begin{bmatrix} Z^{-1} & ? \\ U & ? \end{bmatrix}, G^{-1} := \begin{bmatrix} Y & ? \\ V & ? \end{bmatrix}.$$

Notice that we can always calculate blocks "?" in order to have $GG^{-1} = I$. Also introduce the following:

$$\begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} := \begin{bmatrix} V^T & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} Q & F \\ H & 0 \end{bmatrix} \begin{bmatrix} UZ & 0 \\ 0 & I \end{bmatrix}^{-1}.$$

Let $R := V^T UZ$, and we are ready for the main result:

Theorem 5 *For $\gamma > 0$, the estimation error transfer function $\|T_M\|_\infty^2 < \gamma$ for all*

$M \in \mathcal{M}$, if the following LMI is satisfied

$$\begin{bmatrix} P_i & J_i & Z^T A_i & Z^T A_i & Z^T B_i & 0 \\ * & S_i & Y^T A_i + FC_i + Q & Y^T A_i + FC_i & Y^T B_i + FD_i & 0 \\ * & * & Z + Z^T - P_i & Z^T + Y + R^T - J_i & 0 & L^T - H^T \\ * & * & * & Y + Y^T - S_i & 0 & L^T \\ * & * & * & * & I & 0 \\ * & * & * & * & * & \gamma I \end{bmatrix} \succ 0 \quad (3.4)$$

where matrices $Q \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{l\times n}$, $F \in \mathbb{R}^{n\times r}$, $R,\ Z,\ Y \in \mathbb{R}^{n\times n}$ and $P_i = P_i^T$, $S_i = S_i^T$, $J_i \in \mathbb{R}^{n\times n}$ for $i = 1, \ldots, N$. And the filter parameters are given by

$$A_f = QR^{-1},\ B_f = F,\ C_f = HR^{-1}.$$

The proof of Theorem 5 can be done by mimic that of Theorem 5.1 in [19].

3.3 Example

A numerical example is used to compare the result with a single Lyapunov function approach provided in [18]. A discrete-time uncertain system is given as:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.8 & 0.3 + 0.6\delta \\ 0.05 & 0.7 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} d(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 & \sqrt{2} \end{bmatrix} d(k) \\ z(k) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) \end{aligned}$$

where $|\delta| < 1$ and it time-invariant.

Using the method provided in this chapter, we obtain the following filter:

$$\begin{aligned} A_f &= \begin{bmatrix} -0.2274 & 0.22741 \\ -0.6196 & 0.61960 \end{bmatrix}, \qquad B_f = \begin{bmatrix} -0.0956 \\ -0.0604 \end{bmatrix} \\ C_f &= \begin{bmatrix} -13.8001 & -3.8748 \end{bmatrix}. \end{aligned}$$

This filter achieves the worst case performance 4.20, comparing favorably to 5.00 with the algorithm proposed in [18]. Figure 3.1 shows the performance of these two filters over the uncertainty. We can see that the filter designed with the algorithm proposed

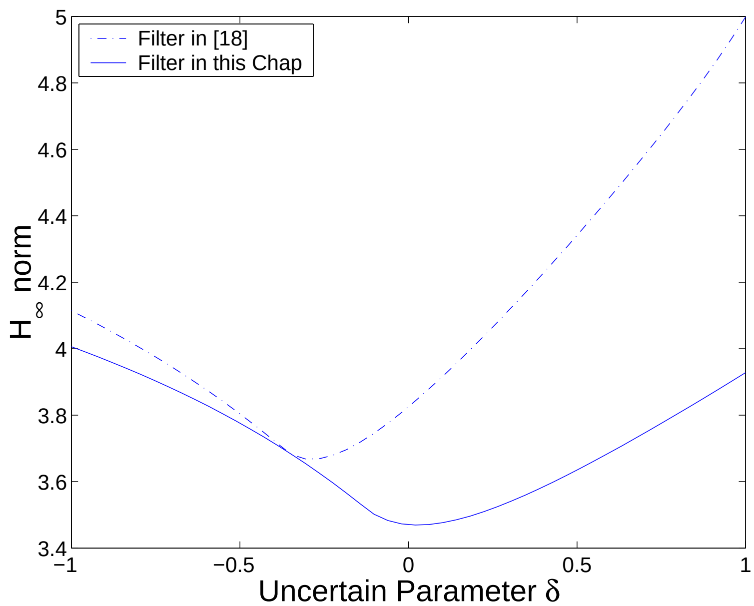

Figure 3.1: Two Robust H_∞ Filters

in the chapter works better in a large range of the uncertainties when we assume the uncertainty is time-invariant. When the uncertainty is time-varying, we can use a single Lyapunov function to find a performance upper bound for the filter in this chapter, and the result is 6.31, which is worse than the one designed by [18].

Another phenomenon worth noticing is that in this example there is still a gap between the worst case performance upper bound 4.20 and the actual worst case performance 4.05.

3.4 Conclusions

In this chapter, we give an LMI formulation to design robust H_∞ filters for time-invariant uncertain systems, based on parameter-dependent Lyapunov functions. This result achieves better worst case performance than approaches with single Lyapunov functions when the uncertain systems is time-invariant. But there is still a gap between the upper bound and actual worst case performance, and in the next chapter we will try to design the optimal robust filter.

Chapter 4

Optimal worst case robust H_∞ filter design – direct method

Results in Chapters 2 and 3, and also most other robust filter results, are characterized by first upper-bounding the performance objective, then selecting filter parameters to minimize the upper bound. There is little quantitative analysis on the conservativeness introduced by the use of upper bounds.

In this chapter, we exploit that the robust filter design problem (with model uncertainty and noise) is convex in the filter as an operator. This fact seems largely unnoticed in the literature. Its implication is that the optimization can be carried out directly, rather than minimizing an upper bound of the objective function. The difficulty is that the optimization problem is infinite dimensional, in both variables and constraints. We show that finite dimensional approximations can be used to obtain suboptimal solutions with any degree of accuracy. A design algorithm is proposed, which consists of a series of finite dimensional approximations. Bounds for approximation errors are calculated for each finite dimensional problem, via worst case analysis and convex optimization. Paganini and Giusto [35] also noticed the convexity of robust synthesis, but their work was on prefilters only. Our work not only studies more general cases, but also validates the use of finite dimensional approximation and gives an algorithm to design a suboptimal filter for any given precision. Some ideas in this chapter follow from the work of Boyd [5], Ghulchak and Rantzer [20] and Dahleh [8].

One of the reasons that the upper bound method is widely used is because the worst case analysis of uncertain systems is difficult. In general, the computation complexity of worst case gain (similar to the calculation of structured singular value μ) is NP hard. There are practical algorithms, see [2], [30], [31] and [61], based on branch & bound approach that can compute it to any specified tolerance. The situation considered in [31] is matrices with real perturbations. Here we will extend those ideas to matrices with additional complex perturbations blocks (unmodelled dynamics). In this chapter, we give a conceptual branch & bound algorithm first, and then a practical one.

This chapter is organized as follows. In Section 4.1, a conceptual worst case analysis algorithm is given. A practical branch & bound algorithm is presented in Section 4.2. In Section 4.3 a robust filtering problem for systems with structured, time invariant uncertainty is formulated. We consider uncertain discrete-time systems, but the results can easily be transformed to continuous-time via the bilinear transform. It is shown in Section 4.4 that for any given $\epsilon > 0$, there exists a finite dimensional relaxation that results in an ϵ-suboptimal solution for the original infinite dimensional problem. In Section 4.5, we give an algorithm to design a suboptimal filter using convex optimization. An example is given in Section 4.7, and conclusions are drawn in Section 4.8.

The notation is standard. $l_2^n(\mathbb{Z}_+)$ denote the n-dimensional square-summable sequences on non-negative integers. $\mathcal{L}_\infty^{m\times n}$ denote the space of all complex-valued m by n matrix functions on the unit circle that are bounded, i.e., if $\hat{R} \in \mathcal{L}_\infty^{m\times n}$, then $\|\hat{R}\|_\infty = \operatorname{ess\,sup}_\theta \bar{\sigma}(\hat{R}(\theta)) < \infty$. The subspace of $\mathcal{L}_\infty^{m\times n}$ that admit analytic continuations in the unit disk is $\mathcal{H}_\infty^{m\times n}$. The space $\mathcal{RH}_\infty^{m\times n}$ is all real-rational functions of $\mathcal{H}_\infty^{m\times n}$. For $A \in \mathbb{C}^{m\times n}$, $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$ denote the maximum and minimum singular value of matrix A. In a normed space $(X, \|\cdot\|)$, for $x \in X$ and $r > 0$, define $\mathrm{B}(x,r) := \{y \in X : \|y - x\| < r\}$. Suppose B and T are linear operators, with B partitioned as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

If T is dimensioned correctly and $I - B_{11}T$ is invertible, define $F_u(B,T) := B_{22} + B_{21}T\,(I - B_{11}T)^{-1}\,B_{12}$.

4.1 Worst-case analysis via branch & bound

In this section, we outline (tersely and implicitly) a conceptual generalized branch & bound algorithm based on sequences of finite coverings of $\mathbf{B}_{\mathbf{\Delta}}$ with decreasing radii, to compute converging upper and lower bounds for worst case performance (H_∞ norm) of a given uncertain system. Actual algorithms, combining upper bounds based on LMIs, lower bounds using problem-specific search methods, and practical branch & bound schemes are given in Section 4.2.

Consider uncertain systems represented in linear fractional form, as in Figure 4.1. Given $S \in \mathcal{RH}_\infty$, an uncertain set $\mathbf{\Delta}$ and $\epsilon > 0$, where $\mathbf{\Delta}$ is defined as

$$\mathbf{\Delta} := \left\{ \text{diag}[\delta_1^r I_{r_1}, \ldots, \delta_N^r I_{r_N}, \delta_1^c I_{k_1}, \ldots, \delta_S^c I_{k_S}, \Delta_1, \ldots, \Delta_F] : \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}, \Delta_i \in \mathbb{C}^{l_i \times l_i} \right\} \tag{4.1}$$

We denote $\{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq 1\}$ by $\mathbf{B}_{\mathbf{\Delta}}$. We want worst-case gain algorithms with the following convergence property: there exist algorithms L_ϵ and U_ϵ (which return real numbers) such that

$$L_\epsilon(S, \mathbf{\Delta}) \leq \max_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|F_u(S, \Delta)\| \leq U_\epsilon(S, \mathbf{\Delta})$$

and $U_\epsilon(S, \mathbf{\Delta}) - L_\epsilon(S, \mathbf{\Delta}) < \epsilon$. Algorithm L_ϵ must also produce a $\Delta^\epsilon \in \mathbf{B}_{\mathbf{\Delta}}$ and $\theta^\epsilon \in [0, 2\pi]$ such that $\bar{\sigma}\left[F_u(S(e^{j\theta^\epsilon}), \Delta^\epsilon)\right] \geq L_\epsilon(S, \mathbf{\Delta})$.

With S fixed, a state-space model of the uncertain relation between d and e is of

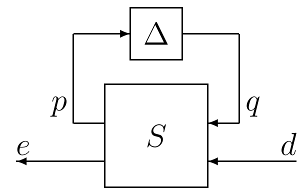

Figure 4.1: General Linear Fractional Form

the form

$$\begin{bmatrix} x_{k+1} \\ p_k \\ e_k \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x_k \\ q_k \\ d_k \end{bmatrix} \qquad q_k = \Delta p_k$$

where the entries $A, B_1, \ldots, D_{22}$ depend on the state-space models of S. We assume that this uncertain system is well-posed (unique solutions to x_{k+1}, p_k, e_k, q_k for any x_k, d_k) and exponentially stable for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$. Define a constant matrix H by the partition implied below

$$\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] := \left[\begin{array}{cc|c} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right]$$

and $\mathbf{\Delta}_e := \{\mathrm{diag}\,[\delta_c I, \Delta] : \delta_c \in \mathbb{C}, \Delta \in \mathbf{\Delta}\}$. Then well-posedness and stability are equivalent to (see [33]) $\det(I - H_{11}\Delta_e) \neq 0,\ \forall \Delta_e \in \mathbf{B}_{\mathbf{\Delta}_e}$ (note: this is equivalent to the spectral radius of A, $\rho(A)$ being less than 1, and $I - \left[D_{11} + C_1\left(e^{j\theta}I - A\right)^{-1} B_1\right]\Delta$ being nonsingular for all $0 \leq \theta \leq 2\pi$, $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$; or, equivalently, $\rho(A) < 1$ and

$\left(I - \left[D_{11} + C_1 (zI - A)^{-1} B_1\right] \Delta\right)^{-1} \in \mathcal{RH}_\infty$ for all $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$). Furthermore,

$$\max_{\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}} \|F_u(G, \Delta)\|_\infty = \max_{\Delta_e \in \mathbf{B}_{\boldsymbol{\Delta}_e}} \bar{\sigma}\left(F_u\left(H, \Delta_e\right)\right)$$

So, the worst case $\mathcal{H}_\infty$ gain of an uncertain dynamic system is the worst case gain of an uncertain constant matrix. Since $\boldsymbol{\Delta}_e$ is also of the general form for $\boldsymbol{\Delta}$ in equation (4.1), we simply focus on a general problem of the form: given a set $\boldsymbol{\Delta}$ as in (4.1), and a complex matrix M, with $I - M_{11}\Delta$ nonsingular for all $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$, develop computable, converging lower and upper bounds for the worst-case gain, $\max_{\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}\left(F_u\left(M, \Delta\right)\right)$. Note that since $\mathbf{B}_{\boldsymbol{\Delta}}$ is compact, it follows that $\left(I - M_{11}\Delta\right)^{-1}$ and $F_u(M, \Delta)$ are bounded on $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$.

4.1.1 Branch & bound algorithm

We outline the necessary continuity arguments which ensure that "branch & bound"-like algorithms will succeed in tightly estimating the worst-case gain. For $c \in \mathbf{B}_{\boldsymbol{\Delta}}$ and $r > 0$, define $Q_{c,r} := \{x : x = c + r\Delta,\ \Delta \in \mathbf{B}_{\boldsymbol{\Delta}}\}$ to denote a ball in $\boldsymbol{\Delta}$ (possibly spilling out of $\mathbf{B}_{\boldsymbol{\Delta}}$) with center c and radius $r > 0$. With M fixed, define $L_{c,r} := \bar{\sigma}\left(F_u(M, c)\right)$ and

$$U_{c,r} := \frac{r}{2} + \inf_{\alpha > 0} \alpha \textit{ subject to } \bar{\sigma}\left(\begin{bmatrix} r^{\frac{1}{2}} M_{11}(I - cM_{11})^{-1} r^{\frac{1}{2}} & r^{\frac{1}{2}} (I - M_{11}c)^{-1} M_{12} \\ \frac{1}{\alpha} M_{21}(I - cM_{11})^{-1} r^{\frac{1}{2}} & \frac{1}{\alpha} F_u(M, c) \end{bmatrix}\right) < 1.$$

Since $(I - M_{11}c)^{-1}$ is bounded on $c \in \mathbf{B_\Delta}$, it follows that for small enough r, $U_{c,r}$ is finite for all $c \in \mathbf{B_\Delta}$, and moreover

$$L_{c,r} \leq \max_{\Delta \in Q_{c,r} \cap \mathbf{B_\Delta}} \bar{\sigma}\left[F_u(M, \Delta)\right] \leq \max_{\Delta \in Q_{c,r}} \bar{\sigma}\left[F_u(M, \Delta)\right] \leq U_{c,r}.$$

To see this, note that if $\alpha > 0$ is feasible in the definition of $U_{c,r}$, then for all $\Delta \in \mathbf{B_\Delta}$ it follows (eg., [33]) that $\bar{\sigma}\left[F_u(M, c + r\Delta)\right] < \alpha$. The bound on $L_{c,r}$ is trivial.

In a branch & bound approach, we cover $\mathbf{B_\Delta}$ with a finite union of these balls, $\mathbf{B_\Delta} \subset \cup_{i=1}^{N} Q_{c_i,r_i}$, and define bounds associated with the particular covering as

$$L := \max_{1 \leq i \leq N} L_{c_i,r_i} \leq \max_{\Delta \in \mathbf{B_\Delta}} \bar{\sigma}\left[F_u(M, \Delta)\right] \leq \max_{1 \leq i \leq N} U_{c_i,r_i} =: U$$

(the element of $\mathbf{B_\Delta}$ that achieves L is simply the c_i value associated with the maximization). If it is true that $U_{c,r} - L_{c,r} \to 0$ uniformly on $c \in \mathbf{B_\Delta}$, as $r \to 0$, then one guarantees ([2]) that L and U will converge as the covering becomes finer (ie., as the covering radii shrink). A proof verifying the uniform convergence of $U_{c,r} - L_{c,r}$ in our situation is given below.

Theorem 6 *Given $\epsilon > 0$, there exists $\delta > 0$, such that if $r < \delta$, $c \in \mathbf{B_\Delta}$, then $U_{c,r} - L_{c,r} < \epsilon$.*

Proof. Since $\bar{\sigma}(F_u(M, c))$ is bounded on $c \in \mathbf{B_\Delta}$, given any $\epsilon > 0$, there exist $\gamma < 1$ such that for all $c \in \mathbf{B_\Delta}$,

$$\bar{\sigma}\left(F_u(M, c)\right) \leq \left[\bar{\sigma}\left(F_u(M, c)\right) + \frac{\epsilon}{2}\right]\gamma = \left(L_{c,r} + \frac{\epsilon}{2}\right)\gamma. \tag{4.2}$$

It is always true that

$$
\begin{aligned}
&\bar{\sigma}\left(\begin{bmatrix} r^{\frac{1}{2}}M_{11}(I-cM_{11})^{-1}r^{\frac{1}{2}} & r^{\frac{1}{2}}(I-M_{11}c)^{-1}M_{12} \\ \frac{1}{L_{c,r}+\frac{\epsilon}{2}}M_{21}(I-cM_{11})^{-1}r^{\frac{1}{2}} & \frac{1}{L_{c,r}+\frac{\epsilon}{2}}F_u(M,c) \end{bmatrix}\right) \\
\leq\ &\bar{\sigma}\left(r^{\frac{1}{2}}M_{11}(I-cM_{11})^{-1}r^{\frac{1}{2}}\right)+\bar{\sigma}\left(r^{\frac{1}{2}}(I-M_{11}c)^{-1}M_{12}\right)+ \\
&\bar{\sigma}\left(\frac{1}{L_{c,r}+\frac{\epsilon}{2}}M_{21}(I-cM_{11})^{-1}r^{\frac{1}{2}}\right)+\bar{\sigma}\left(\frac{1}{L_{c,r}+\frac{\epsilon}{2}}F_u(M,c)\right). \qquad (4.3)
\end{aligned}
$$

Since $\bar{\sigma}((I-M_{11}c)^{-1})$ and $\bar{\sigma}((I-cM_{11})^{-1})$ are bounded on $c \in \mathbf{B_\Delta}$, and $\frac{1}{L_{c,r}+\frac{\epsilon}{2}}$ is bounded on $c \in \mathbf{B_\Delta}$ by $\frac{2}{\epsilon}$, there exists $\delta > 0$ (and without loss of generality, $\delta < \epsilon$), such that if $r \leq \delta$,

$$
\begin{aligned}
&\bar{\sigma}\left(r^{\frac{1}{2}}M_{11}(I-cM_{11})^{-1}r^{\frac{1}{2}}\right)+\bar{\sigma}\left(r^{\frac{1}{2}}(I-M_{11}c)^{-1}M_{12}\right) \\
&\quad +\bar{\sigma}\left(\frac{1}{L_{c,r}+\frac{\epsilon}{2}}M_{21}(I-cM_{11})^{-1}r^{\frac{1}{2}}\right) < 1-\gamma.
\end{aligned}
$$

Substituting above and (4.2) into (4.3), we have

$$
\bar{\sigma}\left(\begin{bmatrix} r^{\frac{1}{2}}M_{11}(I-cM_{11})^{-1}r^{\frac{1}{2}} & r^{\frac{1}{2}}(I-M_{11}c)^{-1}M_{12} \\ \frac{1}{L_{c,r}+\frac{\epsilon}{2}}M_{21}(I-cM_{11})^{-1}r^{\frac{1}{2}} & \frac{1}{L_{c,r}+\frac{\epsilon}{2}}F_u(M,c) \end{bmatrix}\right) < (1-\gamma)+\gamma = 1.
$$

So $L_{c,r} + \frac{\epsilon}{2} \geq U_{c,r} - \frac{r}{2}$, hence $U_{c,r} - L_{c,r} \leq \frac{\epsilon}{2} + \frac{r}{2} < \epsilon$ and the proof is complete. ■

4.2 A practical branch & bound algorithm

The branch & bound algorithm given in last section is difficult to implement since it is hard to divide complex uncertainties. In this section, we give a practical worst case analysis algorithm for problems with any number of real parameters, and 2 (or less) full, unstructured unmodelled dynamics uncertain elements.

In this section, we also propose a method to perform this calculation over frequency. A general method is to perform worst case gain calculation over a frequency grid. With this method, it is difficult to bound the error introduced by this grid, or we have to do calculations on a fine grid. An alternative method is based on transforming frequency to complex scalar perturbation, as shown in last section. We extend this further by transforming this to a compact set on the real line. The benefit gained is that we can calculate the worst case gain using branch & bound method given later, which can be seen as a systemic way of frequency griding.

After transforming frequency to real uncertainties, the worst case performance assessment of an uncertain dynamic system becomes worst case gain calculation of a constant matrix. So we will present this transformation first, then discuss the worst case gain calculation.

4.2.1 Frequency and scalar complex uncertainty

In last section, we show that in discrete-time systems, frequency can be treated as a repeated complex uncertainty. In practice, it is much easier to branch & bound on scalar real uncertainties than complex ones, we transform complex uncertainty to compact sets on the real line. Note the general bilinear transformation maps the upper unit circle to $[0, \infty)$, which is not compact.

Lemma 9 *Consider stable discrete-time systems with state-space representation (A, B, C, D), where $A \in \mathbb{R}^{n \times n}$ and B, C, D are real matrices with appropriate dimen-*

sions. $z \in \mathbb{C}$ *denotes a complex number. Let*

$$H := \begin{bmatrix} \frac{1}{j} I_n & \sqrt{2} I_n \\ \sqrt{2} I_n & j I_n \end{bmatrix} = \begin{bmatrix} \frac{1}{j} & \sqrt{2} \\ \sqrt{2} & j \end{bmatrix} \otimes I_n,$$

where $\otimes$ *is the Kronecker product, and we use* $*$ *denotes star product [31], then the* $\mathcal{H}_\infty$ *norm of this system can be calculated as follows:*

$$\sup_{|z|>1} F_u \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{1}{z} I_n \right) = \max_{d \in [-1,1]} F_u \left(H * \begin{bmatrix} A & B \\ C & D \end{bmatrix}, d I_n \right).$$

Proof. It is easy to know that $f(d) := \frac{dj-1}{j-d}$ maps $[-1,1]$ to upper unit circle on the complex plane. The $\mathcal{H}_\infty$ norm of the system is then

$$\begin{aligned} & \sup_{|z|\geq 1} \bar{\sigma} \left(F_u \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{1}{z} I_n \right) \right) \\ = & \max_{|z|=1} \bar{\sigma} \left(F_u \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{1}{z} I_n \right) \right) = \max_{|z|=1} \bar{\sigma} \left(F_u \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, z I_n \right) \right) \\ = & \max_{d \in [-1,1]} \bar{\sigma} \left(F_u \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{dj-1}{j-d} I_n \right) \right). \end{aligned}$$

In the last equality above, since $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is real, considering upper unit circle is enough. Easy algebra shows that $\frac{dj-1}{j-d} = F_u \left(\begin{bmatrix} \frac{1}{j} & \sqrt{2} \\ \sqrt{2} & j \end{bmatrix}, d \right)$, and we can get the final form above. ■

This result can be extended to worst case performance assessment.

Lemma 10 *Given real matrix M with appropriate dimensions, and structured subspace $\mathbf{\Delta}$ given in (4.1) as well as associated $\mathbf{B}_{\mathbf{\Delta}}$. Suppose for $z \in \mathbb{C}$ and for any $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$, $\sup_{|z|\geq 1}\ \bar{\sigma}\left(F_u\left(F_u\left(M, \frac{1}{z}I_n\right), \Delta\right)\right) < \infty$, then*

$$\sup_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\sup_{|z|\geq 1}\ \bar{\sigma}\left(F_u\left(F_u\left(M, \frac{1}{z}I_n\right), \Delta\right)\right) = \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\max_{d\in[-1,1]}\ \bar{\sigma}\left(F_u(F_u(H * M, dI_n), \Delta)\right).$$

Proof. We know that $\sup_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\sup_{|z|\geq 1}\ \bar{\sigma}\left(F_u\left(F_u\left(M, \frac{1}{z}I_n\right), \Delta\right)\right)$ achieves the supreme at some $\Delta^* \in \mathbf{B}_{\mathbf{\Delta}}$ and $|z^*| = 1$. Since sets $\mathbf{B}_{\mathbf{\Delta}}$ and $\{z \in \mathbb{C} : |z| = 1\}$ contain complex conjugate pairs and since M is real, we can assume that z^* lies on the upper unit circle. The rest of this proof can be obtained by mimic the proof of Lemma 9. ■

After this, worst case gain calculation over frequency reduces to a constant matrix problem with an augmented uncertainty block structure (an additional repeated real). With a branch & bound algorithm given later, it can be calculated to any specified accuracy.

4.2.2 Worst-case performance assessment

The algorithm in this section can only deal with $\mathbf{\Delta}$ that contains real scalar parameters, and 2 (or less) full complex matrices. We also need to treat real and complex uncertainty differently. So we partition $\mathbf{\Delta}$ in (4.1):

$$\mathbf{\Delta} := \{\text{diag}[\Delta_R, \Delta_C] :\ \Delta_R \in \mathbf{\Delta_R},\ \Delta_C \in \mathbf{\Delta_C}\},$$

where $\mathbf{\Delta_R} := \{\text{diag}[\delta_1 I_{k_1}, \ldots, \delta_s I_{k_n}] :\ \delta_i \in \mathbb{R}\} \subset \mathbb{C}^{n_r\times n_r}$, $\mathbf{\Delta_C} := \{\text{diag}[\Delta_1, \Delta_2] : \Delta_j \in \mathbb{C}^{k_{n+j}\times k_{n+j}}\} \subset \mathbb{C}^{n_c\times n_c}$. $\mathbf{B}_R$ and $\mathbf{B_C}$ are similarly defined as $\mathbf{B}_{\mathbf{\Delta}}$. Associated with set

$\mathbf{\Delta_R}$ and $\mathbf{\Delta_C}$ are scaling sets $\mathcal{D}_R$ and $\mathcal{D}_C$, defined as

$$\begin{aligned}\mathcal{D}_R &:= \left\{\text{diag}[D_1, \cdots, D_n] : 0 \prec D_i = D_i^* \in \mathbb{C}^{k_i \times k_i}\right\} \\ \mathcal{D}_C &:= \left\{\text{diag}[d_1 I_{k_{n+1}}, d_2 I_{k_{n+2}}] : \, d_i > 0\right\}.\end{aligned}$$

Complex matrix M is partitioned as follows:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & M_{33} \end{bmatrix}, \tag{4.4}$$

where $M_{11} \in \mathbb{C}^{n_r \times n_r}$, $M_{22} \in \mathbb{C}^{n_c \times n_c}$, and $M_{33} \in \mathbb{C}^{m_p \times n_p}$. The worst case gain is the solution for (or approximately solve) the following problem:

$$\max_{\Delta \in \mathbf{B_\Delta}} \bar{\sigma}(F_u(M, \Delta)). \tag{4.5}$$

Branch & bound algorithm

In a branch & bound algorithm, real uncertainties are divided into cubes. We use $Q_{c,r} \subset \mathbf{B}_R$ to denote a cube with center c and radius r. For a given pair $a \in \mathbb{R}^{k_n}$ and $b \in \mathbb{R}^{k_n}$, with $a_i < b_i$ for each i, denote the cube $Q_{[a,b]} := [a_1 \; b_1] \times [a_2 \; b_2] \times \cdots \times [a_{k_n} \; b_{k_n}] \subset \mathbb{R}^{k_n}$, and diagonal "center" and "radius" matrices are given by

$$c := \begin{bmatrix} \frac{b_1+a_1}{2} I_{k_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{b_{k_n}+a_{k_n}}{2} I_{k_n} \end{bmatrix}, \; r := \begin{bmatrix} \frac{b_1-a_1}{2} I_{k_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{b_{k_n}-a_{k_n}}{2} I_{k_n} \end{bmatrix}.$$

Given M and $Q_{c,r}$, suppose we have easily computable bounds L and U, satisfying

$$L(M,c,r) \leq \max_{\Delta_R \in Q_{c,r}, \Delta_C \in \mathbf{B_C}} \bar{\sigma}\left(F_u\left(M, \begin{bmatrix} \Delta_R & 0 \\ 0 & \Delta_C \end{bmatrix}\right)\right) \leq U(M,c,r),$$

and the lower bound algorithm also yield a perturbation which achieves the lower bound, then the procedure of branch & bound:

1. Initialize list of cubes to the initial cube.

2. Call upper and lower bounds computations on the initial cube.

3. Find cube in ACTIVE list with largest upper bound.

4. Split cube along longest edge into two cubes, compute bounds on both of these new cubes, and replace.

5. If the difference between the largest lower bound and largest upper bound (over current cubes) is less than a specified value, exist; otherwise, make any current cube whose upper bound is lower than another cube's lower bound INACTIVE and go to 3.

The output of this algorithm are the largest lower bound and upper bound, as well as the uncertainty that achieves the lower bound.

In [2], it shows that this branch & bound algorithm terminates in finite number of steps if $U(M,c,r) - L(M,c,r) \to 0$ uniformly (independent of $c \in \mathbf{B}_R$) as $\bar{\sigma}(r) \to 0$. Following sections discuss lower and upper bounds, as well as validating the uniformly convergence condition.

Lower bounds

In this section, we consider two lower bounds. The first one is mainly for analysis convenience, and the second one needs more computation efforts, but in general it is tighter, and results in faster convergence.

The first lower bound of worst case gain on cube $Q_{c,r}$, denoted as β_c, is the worst case gain at the center c of the cube, i.e., $\beta_c := \max_{\Delta_C \in \mathbf{B_C}} \bar{\sigma}\left(F_u(F_u(M,c),\Delta_C)\right)$. Because of the particular structure of this problem, β_c can be calculated by LMIs (see [33] for details):

$$\min_{\beta\in\mathbb{R},\, D\in\mathcal{D}_C} \quad \beta \tag{4.6}$$

$$\text{s.t.} \qquad F_u(M,c)^* \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} F_u(M,c) \prec \begin{bmatrix} D & 0 \\ 0 & \beta^2 I \end{bmatrix}. \tag{4.7}$$

Lemma 11 *β_c is continuous on $c \in \mathbf{B}_R$.*

Above lemma is about the continuity of worst case gain, which follows easily from the following result.

Lemma 12 *For metric spaces X and Y, if $f : X \times Y \to \mathbb{R}$ is continuous, and set Y is compact, then $g(x) := \max_{y\in Y} f(x,y)$ is continuous on X.*

Proof. For any $x \in X$, exists $\hat{y}$ s.t. $f(x,y) = g(x)$. Suppose $x_n \to x$, and since Y is compact, $\exists y_n$ such that $f(x_n,y_n) = g(x_n)$. Need to show $f(x_n,y_n) \to f(x,\hat{y})$ as $n \to \infty$.

Suppose $f(x_n, y_n)$ does not converge to $f(x, \hat{y})$, then there exists $\epsilon > 0$, for any K, exist $n_K > \max(K, n_K - 1)$ such that $|f(x_{n_K}, y_{n_K}) - f(x, \hat{y})| > \epsilon$. For the subsequence $\{y_{n_K}\}$, it has at least one limit point, say, $\bar{y}$, since Y is compact. By construction, $f(x, \bar{y}) \neq f(x, \hat{y})$.

But we still have $x_{n_k} \to x$. Since $f(x, y)$ is continuous, $f(x_{n_k}, y_{n_k}) \to f(x, \bar{y})$. By the definition of y_n, we have $f(x_{n_k}, y_{n_k}) \geq f(x_{n_k}, \hat{y})$, so $f(x, \bar{y}) \geq f(x, \hat{y})$. On the other hand $f(x, \hat{y}) \geq f(x, \bar{y})$ by definition. So we conclude $f(x, \bar{y}) = f(x, \hat{y})$, which leads to a contradiction.

So $f(x_n, y_n) \to f(x, \hat{y})$ as $n \to \infty$, and $g(x)$ is continuous. ■

The second lower bound is given in [31].

LMI upper bound in cubes

In this part, we first give the well-known LMI upper bound for worst case gain calculation with normalized perturbation sets. For upper bounds in cubes, we re-normalize the cube, and then calculate the upper bound.

It is well known that for normalized perturbation, [59], [14], if there exists $D_R \in \mathcal{D}_R$, $D_C \in \mathcal{D}_C$ and $\alpha > 0$ such that

$$M^* \begin{bmatrix} D_R & 0 & 0 \\ 0 & D_C & 0 \\ 0 & 0 & I \end{bmatrix} M \prec \begin{bmatrix} D_R & 0 & 0 \\ 0 & D_C & 0 \\ 0 & 0 & \alpha^2 I \end{bmatrix}, \tag{4.8}$$

then $\max_{\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}(F_u(M, \Delta)) < \alpha$.

To calculate this upper bound on each cube $Q_{c,r}$, we need re-centering/normalizing. Here we restate the result given in [31]. Given $Q_{c,r}$, take $T_{c,r} := \begin{bmatrix} 0 & r^{\frac{1}{2}} \\ r^{\frac{1}{2}} & c \end{bmatrix}$, then

$$\max_{\Delta_R \in Q_{c,r}, \Delta_C \in \mathbf{B_C}} \bar{\sigma}\left(F_u\left(M, \begin{bmatrix} \Delta_R & 0 \\ 0 & \Delta_C \end{bmatrix}\right)\right)$$
$$= \max_{\Delta_R \in \mathbf{B}_R, \Delta_C \in \mathbf{B_C}} \bar{\sigma}\left(F_u\left(T_{c,r} * M, \begin{bmatrix} \Delta_R & 0 \\ 0 & \Delta_C \end{bmatrix}\right)\right),$$

where $T_{c,r} * M = \begin{bmatrix} r^{\frac{1}{2}} M_{11}(I - cM_{11})^{-1} r^{\frac{1}{2}} & r^{\frac{1}{2}}(I - M_{11}c)^{-1}\hat{M}_{12} \\ \hat{M}_{21}(I - cM_{11})^{-1} r^{\frac{1}{2}} & F_u(M, c) \end{bmatrix}$.

The worst case gain upper bound on cube $Q_{c,r}$, i.e., $\max_{\Delta \in \mathbf{B_\Delta}} \bar{\sigma}(F_u(T_{c,r} * M, \Delta))$ can be calculated by (4.8), with $T_{c,r} * M$ substituting M. Denote this upper bound as $\alpha_{c,r}$.

Convergence proof

Recall that to show the branch & bound algorithm converges, we need $\alpha_{c,r} \to \beta_c$ as $r \to 0$ for all $c \in \mathbf{B}_R$. To this end, we have to consider the structured singular value (s.s.v or μ) problem first, and then come back to worst case performance problem. We start with the following lemma:

Lemma 13 $\forall M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, *$\bar{\sigma}(M) \leq \gamma$, for any $\epsilon > 0$, exists $\mathbf{B}_{\mathcal{D}1} := \{d \in$*

$\mathbb{C} : \frac{\epsilon^2}{4\gamma^2} \leq d \leq \frac{4\gamma^2}{\epsilon^2}\}$, *such that*

$$\inf_{d \in \mathbf{B}_{\mathcal{D}1}} \bar{\sigma} \begin{pmatrix} M_{11} & dM_{12} \\ \frac{1}{d} M_{21} & M_{22} \end{pmatrix} \leq \inf_{d \in \mathcal{D}_1} \bar{\sigma} \begin{pmatrix} M_{11} & dM_{12} \\ \frac{1}{d} M_{21} & M_{22} \end{pmatrix} + \epsilon.$$

Proof. Take $\delta_1 = \frac{\epsilon^2}{4\gamma}$, and suppose $\frac{2\gamma}{\epsilon} \geq 1$. First case, if $\bar{\sigma}(M_{12}) \geq \delta_1$ and $\bar{\sigma}(M_{21}) \geq \delta_1$. Suppose the optimal solution is achieved at d^*, then $\bar{\sigma}(d^* M_{12}) \leq \bar{\sigma}(M) \leq \gamma$. Hence $|d^*|\bar{\sigma}(M_{12}) \leq \gamma$, so $|d^*| \leq \frac{\gamma}{\bar{\sigma}(M_{12})} \leq \frac{\gamma}{\delta_1} = \frac{4\gamma^2}{\epsilon^2}$. Similarly, $\frac{1}{|d^*|} \leq \frac{4\gamma^2}{\epsilon^2}$. So $d^* \in \mathbf{B}_{\mathcal{D}1}$.

Second case, without loss of generality, suppose $\bar{\sigma}(M_{12}) \leq \delta_1$, then take $d = \frac{2\gamma}{\epsilon}$, we have

$$\begin{aligned} \bar{\sigma} \begin{pmatrix} M_{11} & dM_{12} \\ \frac{1}{d} M_{21} & M_{22} \end{pmatrix} &= \bar{\sigma} \begin{pmatrix} M_{11} & \frac{2\gamma}{\epsilon} M_{12} \\ \frac{\epsilon}{2\gamma} M_{21} & M_{22} \end{pmatrix} \\ &\leq \bar{\sigma} \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} + \bar{\sigma}\left(\frac{2\gamma}{\epsilon} M_{12}\right) + \bar{\sigma}\left(\frac{\epsilon}{2\gamma} M_{21}\right). \end{aligned}$$

Since $\bar{\sigma}(M_{21}) \leq \bar{\sigma}(M) \leq \gamma$ and $\bar{\sigma}(M_{12}) \leq \delta_1$, we have

$$\begin{aligned} \bar{\sigma} \begin{pmatrix} M_{11} & dM_{12} \\ \frac{1}{d} M_{21} & M_{22} \end{pmatrix} &\leq \bar{\sigma} \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} + \frac{2\gamma}{\epsilon}\delta_1 + \frac{\epsilon}{2\gamma}\gamma \\ &\leq \bar{\sigma} \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &\leq \max\{\bar{\sigma}(M_{11}), \bar{\sigma}(M_{22})\} + \epsilon \\ &\leq \inf_{d \in \mathcal{D}_1} \bar{\sigma} \begin{pmatrix} M_{11} & dM_{12} \\ \frac{1}{d} M_{21} & M_{22} \end{pmatrix} + \epsilon. \end{aligned}$$

Notice here $d = \frac{2\gamma}{\epsilon} \in \mathbf{B}_{\mathcal{D}1}$. Similarly, $\frac{1}{d} \in \mathbf{B}_{\mathcal{D}1}$.

The proof is completed by combining these two cases. ■

Above result can be extended to three-block cases. Let $\mathcal{D}_2 := \{\mathrm{diag}(D, I) : D \in \mathcal{D}_C\}$. Since for any $\gamma > 0$, $(\gamma D)M(\gamma D)^{-1} = DMD^{-1}$, so assuming the third block of $\mathcal{D}_2$ to be identity does not lose any generality.

Lemma 14 *Given M defined in (4.4) with $n_r = 0$, and $\bar{\sigma}(M) \leq \gamma$. For any $\epsilon > 0$, let $\mathbf{B}_{\mathcal{D}2} := \left\{\mathrm{diag}(D, I) : D \in \mathcal{D}_C,\ 81(\frac{\epsilon}{6\gamma})^{34} \leq d_1, d_2 \leq \frac{1}{81}(\frac{6\gamma}{\epsilon})^{34}\right\}$, then*

$$\inf_{D \in \mathbf{B}_{\mathcal{D}2}} \bar{\sigma}\left(D^{\frac{1}{2}} M D^{-\frac{1}{2}}\right) \leq \inf_{D \in \mathcal{D}_2} \bar{\sigma}\left(D^{\frac{1}{2}} M D^{-\frac{1}{2}}\right) + \epsilon.$$

Proof of Lemma 14 may be done in many ways. What we have is similar to the proof of Lemma 13, but much more tedious. It is omitted for simplicity.

Given M in (4.4), $Q_{c,r}$, $\mathcal{D}_2$ and $\mathbf{B}_{\mathcal{D}2}$ in the form as above, define

$$\mu(F_u(M,c)) := \inf_{D \in \mathcal{D}_2} \bar{\sigma}\left(D^{\frac{1}{2}} F_u(M,c) D^{-\frac{1}{2}}\right) \quad (4.9)$$

$$\hat{\mu}(T_{c,r} * M) := \inf_{D_R \in \mathcal{D}_R, D \in \mathbf{B}_{\mathcal{D}2}} \bar{\sigma}\left(\mathrm{diag}(D_R, D)^{\frac{1}{2}} (T_{c,r} * M) \mathrm{diag}(D_R, D)^{-\frac{1}{2}}\right) \quad (4.10)$$

The following result shows that when the cube $Q_{c,r}$ is small enough, the quantity $\mu(F_u(M,c))$ evaluated at the center c can be close to $\hat{\mu}(T_{c,r} * M)$ on the cube.

Theorem 7 *For any $\epsilon > 0$, there exists $\mathbf{B}_{\mathcal{D}2}$ and δ independent of c, such that as $r \leq \delta$,*

$$\hat{\mu}(T_{c,r} * M) \leq \mu(F_u(M,c)) + \epsilon. \quad (4.11)$$

Proof. By Lemma 14, [1] for $\frac{\epsilon}{2} > 0$, there exists bounded set $\mathbf{B}_{\mathcal{D}2} \subset \mathcal{D}_2$, such that $\exists D_C^* \in \mathbf{B}_{\mathcal{D}2}$ and $\bar{\sigma}\left((D_C^*)^{\frac{1}{2}} F_u(M,c)(D_C^*)^{-\frac{1}{2}}\right) \leq \mu(F_u(M,c)) + \frac{\epsilon}{2}$, then,

$$
\begin{aligned}
\hat{\mu}(T_{c,r} * M) &= \inf_{D_R \in \mathcal{D}_R, D \in \mathbf{B}_{\mathcal{D}2}} \bar{\sigma}\left(\begin{bmatrix} D_R^{\frac{1}{2}} & 0 \\ 0 & D^{\frac{1}{2}} \end{bmatrix} T_{c,r} * M \begin{bmatrix} D_R^{-\frac{1}{2}} & 0 \\ 0 & D^{-\frac{1}{2}} \end{bmatrix}\right) \\
&= \inf_{D_R \in \mathcal{D}_R, D \in \mathbf{B}_{\mathcal{D}2}} \\
&\quad \bar{\sigma}\left(\begin{bmatrix} D_R^{\frac{1}{2}} r^{\frac{1}{2}} M_{11}(I - cM_{11})^{-1} r^{\frac{1}{2}} D_R^{-\frac{1}{2}} & D_R^{\frac{1}{2}} r^{\frac{1}{2}} (I - M_{11}c)^{-1} \hat{M}_{12} D^{-\frac{1}{2}} \\ D^{\frac{1}{2}} \hat{M}_{21}(I - cM_{11})^{-1} r^{\frac{1}{2}} D_R^{-\frac{1}{2}} & D^{\frac{1}{2}} F_u(M,c) D^{-\frac{1}{2}} \end{bmatrix}\right) \\
&\leq \inf_{D \in \mathbf{B}_{\mathcal{D}2}} \bar{\sigma}\left(\begin{bmatrix} r^{\frac{1}{2}} M_{11}(I - cM_{11})^{-1} r^{\frac{1}{2}} & r^{\frac{1}{2}} (I - M_{11}c)^{-1} \hat{M}_{12} D^{-\frac{1}{2}} \\ D^{\frac{1}{2}} \hat{M}_{21}(I - cM_{11})^{-1} r^{\frac{1}{2}} & D^{\frac{1}{2}} F_u(M,c) D^{-\frac{1}{2}} \end{bmatrix}\right) \\
&\leq \bar{\sigma}\left(\begin{bmatrix} r^{\frac{1}{2}} M_{11}(I - cM_{11})^{-1} r^{\frac{1}{2}} & r^{\frac{1}{2}} (I - M_{11}c)^{-1} \hat{M}_{12} (D_C^*)^{-\frac{1}{2}} \\ (D_C^*)^{\frac{1}{2}} \hat{M}_{21}(I - cM_{11})^{-1} r^{\frac{1}{2}} & (D_C^*)^{\frac{1}{2}} F_u(M,c) (D_C^*)^{-\frac{1}{2}} \end{bmatrix}\right).
\end{aligned}
$$

Since $(I - cM_{11})^{-1}$ is bounded and $D_C^* \in \mathbf{B}_{\mathcal{D}2}$ for all $c \in \mathbf{B}_R$, there exists δ independent of c, such that when $\bar{\sigma}(r) \leq \delta$,

$$
\hat{\mu}(T_{c,r} * M) \leq \bar{\sigma}\left((D^*)^{\frac{1}{2}} F_u(M,c)(D^*)^{-\frac{1}{2}}\right) + \frac{\epsilon}{2} \leq \mu(F_u(M,c)) + \epsilon.
$$

This completes the proof. ■

Lemma 15 *Recall* $\beta_c = \max_{\Delta_C \in \mathbf{B_C}} \bar{\sigma}\left(F_u(F_u(M,c), \Delta_C)\right)$, *then for any* $c \in \mathbf{B}_R$

$$
\mu\left(\begin{bmatrix} I & 0 \\ 0 & \frac{1}{\beta_c + \epsilon} I \end{bmatrix} F_u(M,c)\right) < 1. \tag{4.12}
$$

Furthermore, left hand side of (4.12) is a continuous function of $c \in \mathbf{B}_R$.

[1] To apply Lemma 14, we need to calculate $\gamma \geq \max \bar{\sigma}_{c \in \mathbf{B}_R}(F_u(M,c))$. It can be shown that after dividing the real uncertainty to small enough cubes, γ can be obtained by solving LMIs, i.e., worst case gain upper bound calculations over theses cubes.

Proof. By the definition of β_c and $\mu(\cdot)$, (4.12) can be obtained according to [33]. To show the left hand side of (4.12) is continuous on $\mathbf{B}_R$, notice that $\mu(\cdot)$ is continuous [34], $\frac{1}{\beta_c+\epsilon}$ is a continuous function on $\mathbf{B}_R$ by Lemma 11, and $F_u(M,c)$ is also a continuous function on $\mathbf{B}_R$ since the LFT is well-posed. ■

We are ready for the main result:

Theorem 8 *For any $\epsilon > 0$, there exists $\mathbf{B}_{\mathcal{D}C}$ and $\delta > 0$ independent of $c \in \mathbf{B}_R$, such that the worst case gain upper bound $\alpha_{c,r}$ and lower bound β_c (over cube $Q_{c,r}$ and $\mathbf{B_C}$) satisfy $\alpha_{c,r} \leq \beta_c + \epsilon$ as $\bar{\sigma}(r) < \delta$.*

Proof. Since left hand side of (4.12) is continuous on compact set $\mathbf{B}_R$, there exists $\gamma_0 < 1$ such that for all $c \in \mathbf{B}_R$,

$$\mu\left(\begin{bmatrix} I_{n_c} & 0 \\ 0 & \frac{1}{\beta_c+\epsilon}I \end{bmatrix} F_u(M,c)\right) < \gamma_0.$$

By Theorem 7, for $\frac{1-\gamma_0}{2} > 0$, there exists δ_1 and $\mathbf{B}_{\mathcal{D}2}$, which are independent of c, such that when $\bar{\sigma}(r) < \delta_1$,

$$\hat{\mu}\left(\begin{bmatrix} I_{n_r+n_c} & 0 \\ 0 & \frac{1}{\beta_c+\epsilon}I \end{bmatrix} T_{c,r} * M\right) \leq \gamma_0 + \frac{1-\gamma_0}{2} < 1 \qquad (4.13)$$

By the definition of $\hat{\mu}$ in (4.11) and [33], (4.13) is true if and only if $\exists\, D_R^* \in \mathcal{D}_R$, $D^* = \mathrm{diag}(D_C^*, I) \in \mathbf{B}_{\mathcal{D}2}$, such that

$$\left(\begin{bmatrix} I & 0 \\ 0 & \frac{1}{\beta_c+\epsilon}I \end{bmatrix}(T_{c,r} * M)\right)^* \begin{bmatrix} D_R^* & 0 \\ 0 & D^* \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & \frac{1}{\beta_c+\epsilon}I \end{bmatrix}(T_{c,r} * M)\right) \prec \begin{bmatrix} D_R^* & 0 \\ 0 & D^* \end{bmatrix}$$

$$(T_{c,r} * M)^* \begin{bmatrix} D_R^* & 0 & 0 \\ 0 & D_C^* & 0 \\ 0 & 0 & \frac{1}{(\beta_c+\epsilon)^2} I \end{bmatrix} (T_{c,r} * M) \prec \begin{bmatrix} D_R^* & 0 & 0 \\ 0 & D_C^* & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$(T_{c,r} * M)^* \begin{bmatrix} (\beta_c+\epsilon)^2 D_R^* & 0 & 0 \\ 0 & (\beta_c+\epsilon)^2 D_C^* & 0 \\ 0 & 0 & I \end{bmatrix} (T_{c,r} * M) \prec (\beta_c+\epsilon)^2 \begin{bmatrix} D_R^* & 0 & 0 \\ 0 & D_C^* & 0 \\ 0 & 0 & I \end{bmatrix}$$

Above inequality is in the form of (4.8). $\mathbf{B}_{\mathcal{D}C}$ can be derived from $\mathbf{B}_{\mathcal{D}2}$, since $\text{diag}(\mathcal{D}_C^*, I) \in \mathbf{B}_{\mathcal{D}2}$, and $(\beta_c + \epsilon)$ is lower bounded by ϵ and upper bounded by the largest upper bound obtained in last iteration. Take $\delta = \delta_1$, then $\alpha_{c,r} < \beta_c + \epsilon$. ■

This convergence proof is mainly for theoretical completeness. In practice, the scaling term $\mathcal{D}_C$ is always bounded in the numerical algorithm.

4.3 Filtering problem formulation and preliminaries

We pose a generalized robust "control" problem that does not, in fact, involve feedback around the designed controller. The absence of feedback simply means that genuine feedback control problems are not addressed by this work, but problems such as robust input design/shaping and robust filtering/estimation are covered. The lack of feedback also means that the resulting minimax optimization is convex (though infinite dimensional). The focus of this chapter is a convergent design algorithm with computable stopping criteria for such problems, as well as the application of the theory to robust $\mathcal{H}_\infty$ linear estimation.

The system considered is shown in Figure 4.2. All signals are multi-variable, though for clarity we will not be overly specific in their dimensions. Nothing is assumed to be scalar, so care in manipulating the non-commuting multi-variable operators will be maintained. For ease of expression, we denote the dimensions of y and u as n_y and n_u, respectively.

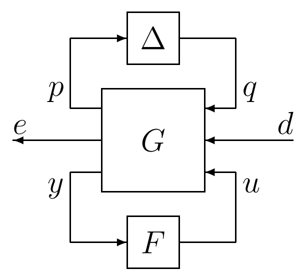

Figure 4.2: General Robust Design Problem

The system consists of a known, stable system G, in feedback with an unknown matrix Δ of the form in equation (4.1) and a to-be-designed linear controller F. There is an obvious 3-by-3 block partition of G into individual relations between its 3 sets of inputs and 3 sets of outputs. With G already assumed stable, it follows that each $G_{ij} \in \mathcal{RH}_\infty$ (of appropriate dimension). We make two main assumptions:

Assumption 1 *For all $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$, $(I - G_{11}\Delta)^{-1}$ is stable, i.e., in $\mathcal{RH}_\infty$.*

Assumption 2 *For all $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$, $G_{33} + G_{31}\Delta\,(I - G_{11}\Delta)^{-1}\,G_{13}$ is* **identically zero**.

The first assumption is standard for most robust filter formulations. The second assumption is the "no feedback around F" assumption, forcing the problem we consider to be an open-loop problem, which accounts for the persistent use of the word "filter" as opposed to "controller." Consequently, the closed-loop map from d to e, denoted $T_{d\to e}(G, \Delta, F)$, is

$$T_{d\to e}(G, \Delta, F) = R_\Delta + U_\Delta F V_\Delta,$$

where

$$\begin{aligned} R_\Delta &:= G_{22} + G_{21}\Delta\left(I - G_{11}\Delta\right)^{-1} G_{12} \\ U_\Delta &:= G_{23} + G_{21}\Delta\left(I - G_{11}\Delta\right)^{-1} G_{13} \\ V_\Delta &:= G_{32} + G_{31}\Delta\left(I - G_{11}\Delta\right)^{-1} G_{12}. \end{aligned}$$

In most cases we use R (and V, U) for simplicity, however to emphasize the dependence on Δ or even frequency, we use R_Δ or $R_{\Delta,\theta}$ (and similar for U and V). Assumption 1 implies that there exist positive constants $\bar{r}, \bar{u}$ and $\bar{v}$ such that for all $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$ and $\theta \in [0\ 2\pi]$, $\bar{\sigma}\left(R_{\Delta,\theta}\right) \leq \bar{r}$, $\bar{\sigma}\left(U_{\Delta,\theta}\right) \leq \bar{u}$ and $\bar{\sigma}\left(V_{\Delta,\theta}\right) \leq \bar{v}$. We also assume that U and V are bounded below,

Assumption 3 *There exists* $\underline{u} > 0$ *and* $\underline{v} > 0$ *such that for some* $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$ *and all* $\theta \in [0\ 2\pi]$, $\underline{\sigma}(U_{\Delta,\theta}) \geq \underline{u}$ *and* $\underline{\sigma}(V_{\Delta,\theta}) \geq \underline{v}$.

An example of a relevant problem without feedback, consider the robust estimation problem depicted in Figure 4.3.

Here, an uncertain plant P_Δ is driven by a generalized disturbance d. Variable y is measured and processed with the intent to estimating z. The estimation error $\hat{z} - z$

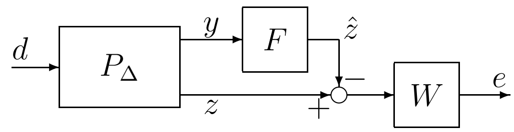

Figure 4.3: Uncertain Plant and Filter

is weighted by a (frequency-dependent) W, giving the error signal which defines the performance objective. A goal of estimation would be to minimize (by choice of F) the worst-case (over all $\Delta \in \mathbf{B_\Delta}$) $\mathcal{H}_\infty$ gain from $d \to e$.

In general, the robust design problem is to minimize, by choice of F, the worst-case (over $\mathbf{B_\Delta}$) gain of $T_{d\to e}(G, \Delta, F)$. In terms of R_Δ, U_Δ and V_Δ, this is

$$\inf_{F \in \mathcal{RH}_\infty} \max_{\Delta \in \mathbf{B_\Delta}} \|R_\Delta + U_\Delta F V_\Delta\|_\infty \tag{4.14}$$

Let Λ_{opt} denote the infimum. Given $\epsilon > 0$, the objective is to find an $F \in \mathcal{RH}_\infty$ whose cost is within ϵ of Λ_{opt}. Such an F is called an ϵ-suboptimal solution.

Definition 2 (ϵ-suboptimal solution) *For $\epsilon > 0$, a given $F \in \mathcal{RH}_\infty$ is an ϵ-*suboptimal solution *for (4.14) if* $\max_{\Delta \in \mathbf{B_\Delta}} \|R_\Delta + U_\Delta F V_\Delta\|_\infty \le \Lambda_{\text{opt}} + \epsilon$.

The uncertainty model in this chapter consists of uncertain real parameters (which are intuitive, and natural to include in an uncertain model) and uncertain complex matrices. Theorems, proofs and algorithms are presented for that set. Nevertheless, some discussion as to the <u>relevance</u> of constant, complex-valued uncertainty is in order. In the case of $\mathcal{H}_\infty$ performance bounds, uncertain complex matrix uncertainty also addresses more realistic unmodelled dynamics uncertainty. In fact, when considering robustness of stability, and robustness of $\mathcal{H}_\infty$ performance, norm-bounded, complex matrix uncertainty is mathematically equivalent to linear, norm bounded, time-invariant real-rational dynamic uncertainty. Roughly, an uncertain system is robustly stable to constant, complex-valued, norm-bounded (by 1, using singular value) uncertain parameters, if and only if it is robustly stable to linear, time-invariant, real-rational, dynamic uncertainty with $\mathcal{H}_\infty$ norm less than 1. Similar statements hold true for robust and worst-case performance as well. Detailed theoretical statements can be found in [53] and references therein.

A important observation is that

$$\max_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|R_\Delta + U_\Delta F V_\Delta\|_\infty$$

is convex on $F \in \mathcal{RH}_\infty$. Indeed, F appears affinely within the norm, so for any fixed uncertainty, the objective is a convex function of F. Furthermore, the point-wise supremum of a family of convex functions is convex ([7]). Therefore, the robust filtering problem in (4.14) is a convex optimization problem.

Though convex, the optimization problem is infinite dimensional: sets $\mathbf{B}_{\mathbf{\Delta}}$ and $[0, 2\pi]$ are infinite; $\mathcal{RH}_\infty$ is infinite dimensional. Practically, we can only obtain suboptimal solutions using finite and finite dimensional approximations. Let $\mathcal{F}_{\mathbf{\Delta}}$ denote a finite subset of $\mathbf{B}_{\mathbf{\Delta}}$, $\mathcal{F}_{\mathbf{\Theta}}$ a finite subset of $[0\ 2\pi]$ and $\mathbf{\Phi} := \{\phi_1, \phi_2, \ldots, \phi_K\}$

a finite, linear independent set in $\mathcal{RH}_\infty$. Define a finite dimensional subspace $\mathcal{Q}_{\mathbf{\Phi}}$

$$\mathcal{Q}_{\mathbf{\Phi}} := \left\{ F : \ F(z) = \sum_{k=1}^{K} Q_k \phi_k(z), \ Q_k \in \mathbb{R}^{n_u \times n_y} \right\}. \tag{4.15}$$

For $\rho > 0$, let $B_\rho \mathcal{Q}_{\mathbf{\Phi}} := \{F \in \mathcal{Q}_{\mathbf{\Phi}} : \|F\|_\infty \leq \rho\}$, and $\mathcal{Q}_{\rho,\mathbf{\Phi}} := \{F \in \mathcal{Q}_{\mathbf{\Phi}} : \bar{\sigma}(Q_k) \leq \rho, \ k = 1, \ldots, K\}$. A finite dimensional approximation of problem (4.14), parameterized by ρ,

$$\Lambda_{\rho,\mathbf{\Phi},\mathcal{F}_{\mathbf{\Delta}},\mathcal{F}_{\mathbf{\Theta}}} := \inf_{F \in \mathcal{Q}_{\rho,\mathbf{\Phi}}} \max_{\Delta \in \mathcal{F}_{\mathbf{\Delta}}} \max_{\theta \in \mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma} \left(R_{\Delta,\theta} + U_{\Delta,\theta} F(\theta) V_{\Delta,\theta} \right). \tag{4.16}$$

Since $\mathcal{Q}_{\rho,\mathbf{\Phi}}$ is compact, and $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$ are finite, the infimum is achieved. Denote a minimizer by $F^*_{\rho,\mathbf{\Phi},\mathcal{F}_{\mathbf{\Delta}}\mathcal{F}_{\mathbf{\Theta}}}$. In Section 4.5.2, we show that the minimization can be carried out with semidefinite programming. For this finite dimensional relaxation to be theoretically useful, we need to show that for any $\epsilon > 0$, we can find ρ, and finite sets $\mathcal{F}_{\mathbf{\Delta}}$, $\mathcal{F}_{\mathbf{\Theta}}$ and $\mathbf{\Phi}$ such that the associated $F^*_{\rho,\mathbf{\Phi},\mathcal{F}_{\mathbf{\Delta}}\mathcal{F}_{\mathbf{\Theta}}}$ is ϵ-suboptimal for (4.14). Furthermore, we need a refinement algorithm (to pick the finite sets) and computable error bounds (to detect ϵ-suboptimality).

For the remainder of the chapter, we always take $\phi_k(z) := z^{k-1}$, and denote $\mathbf{\Phi}$ by its cardinality K, using $\mathcal{Q}_K$, $B_\rho \mathcal{Q}_K$, $\mathcal{Q}_{\rho,K}$, $\Lambda_{\rho,K,\mathcal{F}_{\mathbf{\Delta}},\mathcal{F}_{\mathbf{\Theta}}}$ and $F^*_{\rho,K,\mathcal{F}_{\mathbf{\Delta}},\mathcal{F}_{\mathbf{\Theta}}}$ in place of the general notation above.

Lemma 16 $B_\rho \mathcal{Q}_K \subset \mathcal{Q}_{\rho,K} \subset B_{\sqrt{K}\rho} \mathcal{Q}_K$.

Proof. Let $F \in B_\rho \mathcal{Q}_K$, then $\|F\|_\infty \leq \rho$, hence we have

$$\left(\sum_{k=1}^{K} Q_k e^{j(k-1)\theta} \right) (\cdot)^* \quad \preceq \quad \rho^2 I \qquad \forall \, \theta \in [0, 2\pi]$$

$$\int_0^{2\pi} \left(\sum_{k=1}^{K} Q_k e^{j(k-1)\theta} \right) (\cdot)^* \, d\theta \;\; \preceq \;\; 2\pi\rho^2 I$$

$$\sum_{k=1}^{K} \sum_{l=1}^{K} \int_0^{2\pi} Q_k Q_l^T e^{j(k-l)\theta} \, d\theta \;\; \preceq \;\; 2\pi\rho^2 I.$$

Notice that

$$\int_0^{2\pi} e^{j(k-l)\theta} d\theta = \begin{cases} 2\pi & k = l \\ 0 & k \neq l \end{cases}$$

So we have $\sum_{k=1}^{K} Q_k Q_k^T \preceq \rho^2 I$, hence $Q_k Q_k^T \preceq \rho^2 I$ for $k = 1, \ldots, K$, i.e., $F \in \mathcal{Q}_{\rho,K}$. This proves $B_\rho \mathcal{Q}_K \subset \mathcal{Q}_{\rho,K}$. To show $\mathcal{Q}_{\rho,K} \subset B_{\sqrt{K}\rho} \mathcal{Q}_K$, notice that if $Q_k Q_k^T \preceq \rho^2 I$ for $k = 1, \ldots, K$, then $\sum_{k=1}^{K} Q_k Q_k^T \preceq K\rho^2 I$, hence any $F \in Q_{\rho,K}$ satisfies $F \in B_{\sqrt{K}\rho} \mathcal{Q}_K$. This completes the proof. ■

A continuity result will be used in Section 4.4.

Lemma 17 *The mapping* $\bar{\sigma}\left(R_{(\cdot,\cdot)}\right) : \mathbf{B}_{\mathbf{\Delta}} \times [0, 2\pi] \to \mathbb{C}$ *is uniformly continuous on* $\mathbf{B}_{\mathbf{\Delta}} \times [0, 2\pi]$*, with norm* $\bar{\sigma}(\Delta) + |\theta|$*. Moreover, for fixed* $\rho > 0$ *and* K*, the family of mappings* $\{\bar{\sigma}(U_{(\cdot,\cdot)} F(\cdot) V_{(\cdot,\cdot)}) : F \in \mathcal{Q}_{\rho,K}\}$ *is uniformly equicontinuous.*

Proof. To show $R_{\Delta,\theta}$ is uniformly continuous on $\mathbf{B}_{\mathbf{\Delta}} \times [0, 2\pi]$, it is enough to show that $R_{\Delta,\theta}$ is continuous on this set, since $\mathbf{B}_{\mathbf{\Delta}} \times [0, 2\pi]$ is compact and $\bar{\sigma}(\cdot)$ is a continuous mapping. Recall that $R_{\Delta,\theta} = G_{22}(\theta) + G_{21}(\theta)\Delta \left(I - G_{11}(\theta)\Delta\right)^{-1} G_{12}(\theta)$. By Assumption 1, each individual part of $R_{\Delta,\theta}$ is continuous, so it is continuous.

Similar arguments show that $U_{\Delta,\theta}$ and $V_{\Delta,\theta}$ are uniformly continuous. For given K, we know the family of functions $\{e^{j(k-1)\theta}, k = 1, 2, \ldots, K\}$ is equicontinuous. By definition, for any $F \in \mathcal{Q}_{\rho,K}$, each Q_k is uniformly bounded, so the family of functions

$\mathcal{Q}_{\rho,K}$ is also uniformly equicontinuous. So we conclude that the family of functions $\{U_{\Delta,\theta}F(\theta)V_{\Delta,\theta} : F \in \mathcal{Q}_{\rho,K}\}$ is uniformly equicontinuous. ■

The following simple lemma gives a procedure for generating finite covers for a compact set in a metric space with a given radius. The proof should be easy and hence is omitted.

Lemma 18 *In a metric space X, given any compact set $A \subset X$ and $\delta > 0$. Pick any set $\mathcal{F}_A^0 \subset A$ contains finite number of points. If $\mathcal{F}_A^0$ does not satisfy $A \subset \cup_{x\in\mathcal{F}_A}\mathrm{B}(x,\delta)$, then find any $y \in A$ that $y \notin \cup_{x\in\mathcal{F}_A}\mathrm{B}(x,\delta)$, and update $\mathcal{F}_A^{k+1} := \mathcal{F}_A^k \cup \{y\}$. This procedure stops in finite steps and generates a finite cover of A with balls of radius δ.*

4.4 ϵ-suboptimal filters via finite dimensional relaxation

In this section, we show that the finite dimensional approximations of equation (4.16) can be used to get suboptimal solutions of (4.14). First, modify the original problem in (4.14) to only consider filters in $\mathcal{Q}_K$,

$$\Lambda_K := \inf_{F\in\mathcal{Q}_K} \max_{\Delta\in\mathbf{B_\Delta}} \|R_\Delta + U_\Delta F V_\Delta\|_\infty \tag{4.17}$$

An ϵ-suboptimal solution to (4.17) is any $F \in \mathcal{Q}_K$ which achieves a cost within ϵ of the infimum, (just as Definition 2 defines suboptimal solutions to (4.14)).

Theorem 9 *The infimum for Λ_K is achieved.*

Proof. If $F \in \mathcal{Q}_K$ satisfies $\|F\|_\infty > \frac{2\bar{r}}{\underline{uv}}$, then using the lower bounds on U and V and the upper bound on R (assumptions 3 and 1), we have

$$\max_{\Delta \in \mathbf{B}_\Delta} \|R_\Delta + U_\Delta F V_\Delta\|_\infty > \max_{\Delta \in \mathbf{B}_\Delta} \|R_\Delta\|_\infty$$

and hence such an F is "worse" than simply taking $F = 0$. Consequently, for all $\rho > \frac{2\bar{r}}{\underline{uv}}$,

$$\Lambda_K = \inf_{F \in B_\rho \mathcal{Q}_K} \max_{\Delta \in \mathbf{B}_\Delta} \|R_\Delta + U_\Delta F V_\Delta\|_\infty \tag{4.18}$$

Therefore, with $B_\rho \mathcal{Q}_K$ compact, it follows that the infimum is achieved. ■

Denote any such minimizer by $F_K^* \in B_\rho \mathcal{Q}_K$. Next we show that, given $\epsilon > 0$, there exists $K_0 < \infty$, such that when $K > K_0$, F_K^* is an ϵ-suboptimal solution for (4.14). To do this, recall a classic problem of uniformly approximating functions in $\mathcal{H}_\infty$ by functions in $\mathcal{Q}_K$.

Theorem 10 *(from [47, 43]) Suppose $r > 1$ and $\eta > 0$ are given. Let $U := \{z : |z| < r\} \subset \mathbb{C}$ denote an open disk. For any X that is analytic on U, and $\sup_{z \in U} \bar{\sigma}(X(z)) \leq \eta$, then $\inf_{F \in \mathcal{Q}_K} \|X - F\|_\infty \leq \frac{\eta}{r^K}$.*

Using this result, we can obtain an error bound for the robust filtering by using $\mathcal{Q}_K$ rather than $\mathcal{RH}_\infty$:

Theorem 11 *For every K, $\Lambda_{\text{opt}} \leq \Lambda_K$. Moreover $\Lambda_K \to \Lambda_{\text{opt}}$ as $K \to \infty$.*

Proof. $\Lambda_{\text{opt}} \leq \Lambda_K$ is trivial because $\mathcal{RH}_\infty \supset \mathcal{Q}_K$. For any $\epsilon > 0$, let $F^0 \in \mathcal{RH}_\infty$ be an $\frac{\epsilon}{2}$-suboptimal solution for (4.14). Since $F^0 \in \mathcal{RH}_\infty$, it has only finitely many poles outside the unit disk $D := \{z : |z| < 1\}$. Hence there exists $r > 1$ and $\eta > 0$ such that F^0 is analytic on rD, and bounded by η. Choose K_0 so that $\frac{\eta}{r^{K_0}} < \frac{\epsilon}{4\bar{u}\bar{v}}$. Then for any $K \geq K_0$, by Theorem 10, there exists $F_K \in \mathcal{Q}_K$, such that $\|F^0 - F_K\|_\infty < \frac{\epsilon}{4\bar{u}\bar{v}}$. So, for any $\Delta \in \mathbf{B_\Delta}$, we have

$$\begin{aligned}
\|R_\Delta + U_\Delta F_K V_\Delta\|_\infty &= \left\|R_\Delta + U_\Delta F^0 V_\Delta + U_\Delta \left(F_K - F^0\right) V_\Delta\right\|_\infty \\
&\leq \left\|R_\Delta + U_\Delta F^0 V_\Delta\right\|_\infty + \left\|U_\Delta \left(F_K - F^0\right) V_\Delta\right\|_\infty \\
&\leq \left(\Lambda_{\text{opt}} + \frac{\epsilon}{2}\right) + \bar{u}\bar{v} \left\|F_K - F^0\right\|_\infty \\
&\leq \Lambda_{\text{opt}} + \frac{3}{4}\epsilon.
\end{aligned}$$

This is true for all $\Delta \in \mathbf{B_\Delta}$, so $\Lambda_K \leq \max_{\Delta \in \mathbf{B_\Delta}} \|R_\Delta + U_\Delta F_K V_\Delta\|_\infty < \Lambda_{\text{opt}} + \epsilon$ as $K \geq K_0$. Since ϵ is arbitrary, the result follows. ■

Next, we show that given ϵ and K, we can find $\rho > 0$ and finite sets $\mathcal{F}_{\mathbf{\Delta}} \subset \mathbf{B_\Delta}$ and $\mathcal{F}_{\mathbf{\Theta}} \subset [0, 2\pi]$, such that $F^*_{\rho,K,\mathcal{F}_{\mathbf{\Delta}}\mathcal{F}_{\mathbf{\Theta}}}$ is an ϵ-suboptimal solution for (4.17). For simplicity, let $F^*_{K,\mathcal{F}}$ denote $F^*_{\rho,K,\mathcal{F}_{\mathbf{\Delta}}\mathcal{F}_{\mathbf{\Theta}}}$ and $\Lambda_{K,\mathcal{F}}$ denote $\Lambda_{\rho,K,\mathcal{F}_{\mathbf{\Delta}}\mathcal{F}_{\mathbf{\Theta}}}$.

Theorem 12 *For any $\epsilon > 0$ and $K > 0$, there exists $\rho > 0$ and finite sets $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$ such that*

$$\max_{\Delta \in \mathbf{B_\Delta}} \left\|R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta\right\|_\infty < \Lambda_{K,\mathcal{F}} + \epsilon.$$

Remark 4 *Additional inequalities (which always hold) are useful to record.*

$$\Lambda_{K,\mathcal{F}} \leq \Lambda_K \leq \underbrace{\max_{\Delta \in \mathbf{B_\Delta}} \left\|R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta\right\|_\infty < \Lambda_{K,\mathcal{F}} + \epsilon}_{\text{Theorem 12}} \leq \Lambda_K + \epsilon$$

(to show $\Lambda_{K,\mathcal{F}} \leq \Lambda_K$, notice that $F_K^* \in B_\rho \mathcal{Q}_K$, and by Lemma 16, $\mathcal{Q}_{\rho,K} \supset B_\rho \mathcal{Q}_K$, furthermore, fewer constraints are used in obtaining $\Lambda_{K,\mathcal{F}}$).

Proof. Choose $\rho > \frac{2\bar{r}}{\underline{uv}}$. By Lemma 17, there exists $\delta_1 > 0$, such that if $\|(\Delta_1, \theta_1) - (\Delta_2, \theta_2)\| < \delta_1$, then $\bar{\sigma}\left(R_{\Delta_1,\theta_1} - R_{\Delta_2,\theta_2}\right) < \epsilon/2$. Similarly, there exists $\delta_2 > 0$, such that when $\|(\Delta_1, \theta_1) - (\Delta_2, \theta_2)\| < \delta_2$, then $\bar{\sigma}\left(U_{\Delta_1,\theta_1} F(\theta_1) V_{\Delta_1,\theta_1} - U_{\Delta_2,\theta_2} F(\theta_2) V_{\Delta_2,\theta_2}\right) < \epsilon/2$ for all $F \in \mathcal{Q}_{\rho,K}$. Take $\delta := \min(\delta_1, \delta_2)$. Consider any finite sets $\mathcal{F}_{\boldsymbol{\Delta}} = \{\Delta_i\}_{i=1}^M \subset \mathbf{B}_{\boldsymbol{\Delta}}$, and $\mathcal{F}_{\boldsymbol{\Theta}} = \{\theta_j\}_{j=1}^N \subset [0, 2\pi]$. Clearly

$$\max_{\Delta \in \mathcal{F}_{\boldsymbol{\Delta}}} \max_{\theta \in \mathcal{F}_{\boldsymbol{\Theta}}} \bar{\sigma}(R_{\Delta,\theta} + U_{\Delta,\theta} F(\theta) V_{\Delta,\theta}) \leq \max_{\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}} \|R_\Delta + U_\Delta F V_\Delta\|_\infty .$$

Assume further that $\frac{\delta}{2}$-balls around these finite sets cover $\mathbf{B}_{\boldsymbol{\Delta}}$ and $[0\ 2\pi]$,

$$\mathbf{B}_{\boldsymbol{\Delta}} \subset \bigcup_{1 \leq i \leq M} \mathrm{B}(\Delta_i, \delta/2), \qquad [0, 2\pi] \subset \bigcup_{1 \leq j \leq N} \mathrm{B}(\theta_j, \delta/2)$$

(this is always possible since $\mathbf{B}_{\boldsymbol{\Delta}}$ and $[0, 2\pi]$ are compact). Then, for every $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$ and every $\theta \in [0\ 2\pi]$, there exist $\Delta_i \in \mathcal{F}_{\boldsymbol{\Delta}}, \theta_j \in \mathcal{F}_{\boldsymbol{\Theta}}$ such that $\Delta \in \mathrm{B}(\Delta_i, \frac{\delta}{2})$ and $\theta \in \mathrm{B}(\theta_j, \frac{\delta}{2})$. It follows that for all $F \in \mathcal{Q}_{\rho,K}$,

$$\begin{aligned}
&\bar{\sigma}\left[R_{\Delta,\theta} + U_{\Delta,\theta} F(\theta) V_{\Delta,\theta} - R_{\Delta_i,\theta_j} - U_{\Delta_i,\theta_j} F(\theta_j) V_{\Delta_i,\theta_j}\right] \\
&\qquad = \bar{\sigma}\left[R_{\Delta,\theta} - R_{\Delta_i,\theta_j} + U_{\Delta,\theta} F(\theta) V_{\Delta,\theta} - U_{\Delta_i,\theta_j} F(\theta_j) V_{\Delta_i,\theta_j}\right] \\
&\qquad \leq \bar{\sigma}\left[R_{\Delta,\theta} - R_{\Delta_i,\theta_j}\right] + \bar{\sigma}\left[U_{\Delta,\theta} F(\theta) V_{\Delta,\theta} - U_{\Delta_i,\theta_j} F(\theta_j) V_{\Delta_i,\theta_j}\right] \\
&\qquad < \epsilon.
\end{aligned} \tag{4.19}$$

Summarizing (and rearranging) – for every $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$ and $\theta \in [0\ 2\pi]$, there exist $\Delta_i \in \mathcal{F}_{\boldsymbol{\Delta}}, \theta_j \in \mathcal{F}_{\boldsymbol{\Theta}}$ such that for all $F \in \mathcal{Q}_{\rho,K}$

$$\bar{\sigma}\left[R_{\Delta,\theta} + U_{\Delta,\theta} F(\theta) V_{\Delta,\theta}\right] < \bar{\sigma}\left[R_{\Delta_i,\theta_j} + U_{\Delta_i,\theta_j} F(\theta_j) V_{\Delta_i,\theta_j}\right] + \epsilon.$$

Maximizing over each set yields that for all $F \in \mathcal{Q}_{\rho,K}$

$$\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \|R_\Delta + U_\Delta F V_\Delta\|_\infty < \max_{\Delta\in\mathcal{F}_{\mathbf{\Delta}}} \max_{\theta\in\mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma}\left[R_{\Delta,\theta} + U_{\Delta,\theta}F(\theta)V_{\Delta,\theta}\right] + \epsilon. \tag{4.20}$$

Use $F = F^*_{K,\mathcal{F}}$ to obtain the result. ■ To conclude the section, combine Theorem 11 and 12, along with the proof of Theorem 12, yielding

Theorem 13 *For any $\epsilon > 0$, there exists $\rho > 0$ and $K_0 < \infty$, and for any $K > K_0$, there exists $\delta > 0$, such that for any finite sets $\mathcal{F}_{\mathbf{\Delta}} \subset \mathbf{B}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}} \subset [0, 2\pi]$ satisfying $\cup_{\Delta\in\mathcal{F}_{\mathbf{\Delta}}}\mathrm{B}(\Delta,\delta) \supset \mathbf{B}_{\mathbf{\Delta}}$ and $\cup_{\theta\in\mathcal{F}_{\mathbf{\Theta}}}\mathrm{B}(\theta,\delta) \supset [0, 2\pi]$, it follows that*

$$\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \|R + UF^*_{K,\mathcal{F}}V\|_\infty \leq \Lambda_{\text{opt}} + \epsilon.$$

Based on ϵ, this gives a criterion for choosing filter order K and finite sets $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$ such that $F^*_{K,\mathcal{F}}$ is ϵ-suboptimal. In practice though, this criterion is very conservative (regarding δ and K_0) and not constructive. In Section 4.5, we propose a successive finite dimensional approximation to carry out the design.

4.5 Design algorithm and error bounds

In this section, a successive design procedure is proposed, and sets $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$ will be refined in each iteration. In this process, we also compute bounds for approximation error to determine whether to stop. Similar to Section 4.4, the bounds are calculated in two steps. The error bound related to the use of $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$ is calculated via worst case analysis, and it converges to zero. An overall error bound is then calculated via convex optimization.

4.5.1 An algorithm

The basic idea is to start with $\mathcal{F}_{\mathbf{\Delta}}$, $\mathcal{F}_{\mathbf{\Theta}}$ and K small, and design a robust filter by solving a finite dimensional optimization problem. Then check the error bounds. If they are close to zero, or the filter order is larger than a prescribed number, stop. Otherwise, add some points to $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$ or increase the filter order K, and design the filter again. The algorithm is as follows:

Algorithm 1 (Robust filter design algorithm)

Initialization: *Pick $\epsilon > 0$ as the desired suboptimal tolerance; set $\rho > \frac{2\bar{r}}{\underline{uv}}$; pick K^* as the highest controller order in the design; pick finite sets $\mathcal{F}_{\mathbf{\Delta}} \subset \mathbf{B}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}} \subset [0, 2\pi]$, and FIR order K;*

Step 1: *Solve problem (4.16) with current $\mathcal{F}_{\mathbf{\Delta}}$, $\mathcal{F}_{\mathbf{\Theta}}$ and K, yields $F^*_{K,\mathcal{F}}$;*

Step 2: *Apply worst case gain analysis on $R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta$ with a tolerance of $\epsilon_0 := \frac{\epsilon}{4}$. Let $\gamma_{K,\mathcal{F}} := U_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right)$, and define $\Gamma := \gamma_{K,\mathcal{F}} - \Lambda_{K,\mathcal{F}}$. If $\Gamma \leq \epsilon/2$, go to Step 3; otherwise, add the points (from L_{ϵ_0}) Δ^{ϵ_0} and θ^{ϵ_0} to sets $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$, and repeat Step 1;*

Step 3: *Calculate an error bound Ψ given in Section 4.5.4. If $\Psi \leq \epsilon$ or $K \geq K^*$, exit; otherwise, increase K, eg., $K = K + 1$, and go back to Step 1.*

Note that the iteration in Steps 1 and 2 ultimately yields an $\epsilon/2$ suboptimal K'th order FIR robust filter. The outer iteration, in Step 3, increases the filter's order until it is effectively "large" enough.

This algorithm can be thought of as a generalization of the cutting plane method, see [44] and [38]. The explicit formulation of Step 1 as a finite-dimensional SDP, and the calculation and convergence of the bounds in Steps 2 and 3 are addressed in the following subsections.

4.5.2 Casting the finite-dimensional problem as an SDP

The finite dimensional problem of equation (4.16) is convex, and can be formulated as a semidefinite program, [6], using standard linear algebra to convert $\bar{\sigma}$ constraints into matrix definiteness constraints. The finite dimensional space $\mathcal{Q}_K$ is defined in (4.15). Let $Q^K := \{Q_1, \ldots, Q_K\} \in \mathbb{R}^{K \times n_u \times n_y}$ denote the "vector" of coefficients. Enumerate the elements of $\mathcal{F}_{\mathbf{\Delta}}$ as $\{\Delta_1, \ldots, \Delta_M\}$ and the elements of $\mathcal{F}_{\mathbf{\Theta}}$ as $\{\theta_1, \ldots, \theta_N\}$. Problem 4.16 can be written as an SDP,

$$\begin{aligned} \min_{t, Q^K \in \mathbb{R}} \quad & t \\ \text{s.t.} \quad & G_{mn}(t, Q^K) \succeq 0 \quad \forall\, m = 1, 2, \ldots, M \text{ and } n = 1, 2, \ldots, N \\ & \begin{bmatrix} \rho I & Q_k^T \\ \mathcal{Q}_k & \rho I \end{bmatrix} \succeq 0 \quad \forall\, k = 1, 2, \ldots, K. \end{aligned} \tag{4.21}$$

The matrix $G_{mn}(t, Q^K)$ is given by

$$G_{mn}(t, Q^K) := \begin{bmatrix} tI & P_{mn}(Q^K)^* \\ P_{mn}(Q^K) & tI \end{bmatrix}$$

where P_{mn} is

$$P_{mn}(Q^K) := R_{\Delta_m, \theta_n} + U_{\Delta_m, \theta_n} \left(\sum_{k=1}^{K} Q_k \phi_k(\theta_n) \right) V_{\Delta_m, \theta_n}.$$

Solving (4.21) yields t^*, and we have $\Lambda_{\rho,K,\mathcal{F}_{\mathbf{\Delta}},\mathcal{F}_{\mathbf{\Theta}}} = t^*$.

Remark 5 *This conversion of an affine $\bar{\sigma}(\cdot)$ minimization into an SDP is common in robust control. The SDP in equation (4.21) is readily solved ([54]).*

4.5.3 Bounds via worst case analysis

We must verify that the stopping criteria of Step 1 and 2 will eventually be met.

Lemma 19 *For any $\epsilon > 0$, the iteration in Steps 1-2 of Algorithm 1 will terminate in a finite number of iterations.*

Proof. We need to show $\Gamma \leq \frac{\epsilon}{2}$ in finite iterations as steps 1-2 of Algorithm 1 are repeated. Notice that (dropping arguments for clarity) $\Gamma = (\gamma_{K,\mathcal{F}} - L_{\epsilon_0}) + (L_{\epsilon_0} - \Lambda_{K,\mathcal{F}})$. The first term $\left(U_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) - L_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right)\right) < \epsilon_0 = \frac{\epsilon}{4}$ follows from worst case gain calculation.

Hence, we will show $\left|L_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) - \Lambda_{K,\mathcal{F}}\right| < \frac{\epsilon}{4}$ in finite iterations when repeating Steps 1-2. It is enough to show $L_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) - \Lambda_{K,\mathcal{F}} < \frac{\epsilon}{4}$ in finite iterations, since $\Lambda_{K,\mathcal{F}} \leq \gamma_{K,\mathcal{F}} < L_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) + \frac{\epsilon}{4}$ is always true. By Lemma 17, for $\frac{\epsilon}{8}$, there exists δ, such that if $\|(\Delta_1, \theta_1) - (\Delta_2, \theta_2)\| < \delta$, then $\bar{\sigma}\left(R_{\Delta_1,\theta_1} - R_{\Delta_2,\theta_2}\right) < \frac{\epsilon}{8}$ and $\bar{\sigma}\left(U_{\Delta_1,\theta_1} F(\theta_1) V_{\Delta_1,\theta_1} - U_{\Delta_2,\theta_2} F(\theta_2) V_{\Delta_2,\theta_2}\right) < \frac{\epsilon}{8}$ for all $F \in \mathcal{Q}_{\rho,K}$. If $(\Delta^{\epsilon_0}, \theta^{\epsilon_0})$ generated by L_{ϵ_0} satisfies $\Delta^{\epsilon_0} \in \bigcup_{\Delta \in \mathcal{F}_{\mathbf{\Delta}}} \mathrm{B}(\Delta, \delta/2)$, and $\theta^{\epsilon_0} \in \bigcup_{\theta \in \mathcal{F}_{\mathbf{\Theta}}} \mathrm{B}(\theta, \delta/2)$, then

$$L_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) - \Lambda_{K,\mathcal{F}}$$

$$\leq \bar{\sigma}\left(R_{\Delta^{\epsilon_0},\theta^{\epsilon_0}} + U_{\Delta^{\epsilon_0},\theta^{\epsilon_0}} F^*_{K,\mathcal{F}}(\theta^{\epsilon_0}) V_{\Delta^{\epsilon_0},\theta^{\epsilon_0}}\right) - \Lambda_{K,\mathcal{F}}$$
$$\leq \frac{\epsilon}{4},$$

the iteration terminates. Conversely, if $L_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) - \Lambda_{K,\mathcal{F}} > \epsilon/4$, then either $\Delta^{\epsilon_0} \notin \bigcup_{\Delta \in \mathcal{F}_{\mathbf{\Delta}}} \mathrm{B}(\Delta, \delta/2)$, or $\theta^{\epsilon_0} \notin \bigcup_{\theta \in \mathcal{F}_{\mathbf{\Theta}}} \mathrm{B}(\theta, \delta/2)$. Without loss of generality, suppose the first case, then $\bigcup_{\Delta \in \mathcal{F}_{\mathbf{\Delta}}} \mathrm{B}(\Delta, \delta/2)$ does not cover $\mathbf{B}_{\mathbf{\Delta}}$, and $\Delta^{\epsilon_0} \in \mathbf{B}_{\mathbf{\Delta}}$. Because $\mathbf{B}_{\mathbf{\Delta}}$ is compact, by Lemma 18, Steps 1 and 2 will generate a finite cover of $\mathbf{B}_{\mathbf{\Delta}}$. Similarly, we can get another finite cover of $[0, 2\pi]$. So we know this procedure guarantees $L_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) - \Lambda_{K,\mathcal{F}} < \frac{\epsilon}{4}$ in finite iterations, and the proof is complete. ■

Remark 6 *In this proof, we show that the stopping criterion of Steps 1 and 2 are met by a continuity argument and that this procedure generates finite covers of uncertainty set* $\mathbf{B}_{\mathbf{\Delta}}$ *and frequency* $[0, 2\pi]$ *for any given radius. Notice that this is a* sufficient *condition, and for the criterion to be met, the actual* $\mathcal{F}_{\mathbf{\Delta}}$ *and* $\mathcal{F}_{\mathbf{\Theta}}$ *can be really small.*

4.5.4 A lower bound

A lower bound of Λ_{opt}, denoted as $\eta_{\mathcal{F}}$, can be obtained if we only consider sets $\mathcal{F}_{\mathbf{\Delta}}$ and $\mathcal{F}_{\mathbf{\Theta}}$:

$$\eta_{\mathcal{F}} := \lim_{K\to\infty} \inf_{F \in \mathcal{Q}_K} \max_{\Delta \in \mathcal{F}_{\mathbf{\Delta}}} \max_{\theta \in \mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma}(R_{\Delta,\theta} + U_{\Delta,\theta} F(\theta) V_{\Delta,\theta}) \tag{4.22}$$
$$\leq \lim_{K\to\infty} \underbrace{\inf_{F \in \mathcal{Q}_K} \max_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|R_\Delta + U_\Delta F V_\Delta\|_\infty}_{\Lambda_K} = \Lambda_{\text{opt}} \tag{4.23}$$

The definition of $\eta_{\mathcal{F}}$ in (4.22) is a limit of results from a family of optimizations, and is hard to compute. But for a special choice of $\mathcal{F}_{\mathbf{\Theta}}$, $\eta_{\mathcal{F}}$ can be computed:

Lemma 20 *If the frequency grid $\mathcal{F}_{\mathbf{\Theta}}$ is uniform, i.e.,* $\left\{\theta_n = \frac{2\pi(n-1)}{N}\right\}_1^N$, *then*

$$\eta_{\mathcal{F}} = \inf_{F\in\mathcal{Q}_N} \max_{\Delta\in\mathcal{F}_{\mathbf{\Delta}}} \max_{\theta\in\mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma}\left[R_{\Delta,\theta} + U_{\Delta,\theta}F(\theta)V_{\Delta,\theta}\right]. \tag{4.24}$$

Proof. It is enough to show that if $\mathcal{F}_{\mathbf{\Theta}}$ is uniform with N points, then for every $K \geq N$,

$$\begin{aligned} &\inf_{F\in\mathcal{Q}_K} \max_{\Delta\in\mathcal{F}_{\mathbf{\Delta}}} \max_{\theta\in\mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma}\left[R_{\Delta,\theta} + U_{\Delta,\theta}F(\theta)V_{\Delta,\theta}\right] = \\ &\quad \inf_{F\in\mathcal{Q}_N} \max_{\Delta\in\mathcal{F}_{\mathbf{\Delta}}} \max_{\theta\in\mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma}\left[R_{\Delta,\theta} + U_{\Delta,\theta}F(\theta)V_{\Delta,\theta}\right]. \end{aligned} \tag{4.25}$$

This can be seen as follows. Suppose $k > N$, then write $k-1 = pN + q$, for integers $p > 0$, and $q \geq 0$ and $q < N$, then we have

$$e^{j\frac{2\pi(n-1)}{N}(k-1)} = e^{j\frac{2\pi(n-1)}{N}(pN+q)} = e^{j\frac{2n\pi(n-1)}{N}q}$$

Notice that the left hand side of (4.25) is

$$\inf_{Q^K} \max_{\Delta\in\mathcal{F}_{\mathbf{\Delta}}} \max_{\theta\in\mathcal{F}_{\mathbf{\Theta}}} \bar{\sigma}\left[R_{\Delta,\theta} + U_{\Delta,\theta}\left(\sum_{k=1}^{K} Q_k e^{j(k-1)\theta}\right)V_{\Delta,\theta}\right],$$

hence only N variables are needed, and $\eta_{\mathcal{F}}$ can be computed. ■

Now, we have the following chain of inequalities:

$$\begin{aligned} \eta_{\mathcal{F}} &\leq \Lambda_{\text{opt}} \\ &\leq \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \|R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta\|_\infty \\ &\leq U_{\epsilon_0}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) = \gamma_{K,\mathcal{F}} \end{aligned}$$

Let $\Psi := \gamma_{K,\mathcal{F}} - \eta_{\mathcal{F}}$. If $\Psi < \epsilon$, then $F^*_{K,\mathcal{F}}$ is ϵ-suboptimal for (4.14), and therefore Ψ serves as a stopping criteria for Algorithm 1.

Since the resulting $\mathcal{F}_{\mathbf{\Theta}}$ from Step 2 of Algorithm 1 might not be uniform, we have to pick another uniform frequency grid ${\mathcal{F}_{\mathbf{\Theta}}}'$ to calculate $\eta_{\mathcal{F}}$. Let ${\mathcal{F}_{\mathbf{\Theta}}}'$ contains K points, and this lower bound given as follows:

$$\eta_{\mathcal{F}} = \inf_{F \in \mathcal{Q}_K} \max_{\Delta \in \mathcal{F}_{\mathbf{\Delta}}} \max_{\theta \in {\mathcal{F}_{\mathbf{\Theta}}}'} \bar{\sigma}(R_{\Delta,\theta} + U_{\Delta,\theta} F(\theta) V_{\Delta,\theta}).$$

The reason that we have to specify K^* in this algorithm is because that we do not have the convergence proof that if $K \to \infty$, then $\Psi \to 0$.

We summarize the properties of Algorithm 1 as follows:

- For any given filter order K and tolerance $\epsilon > 0$, Steps 1 and 2 obtain the ϵ-suboptimal filter or order K.

- Since $\Lambda_K \to \Lambda_{\text{opt}}$ as $K \to \infty$, we know we can achieve the optimal filter as K increases (set K^* to be infinity).

- A computable lower bound $\eta_{\mathcal{F}}$ is given, but currently, we can not show the convergence of the algorithm with this lower bound. In other words, we do not have a guaranteed stopping criteria.

4.6 Suboptimal filters

The filter design algorithm in Section 4.5 relies on converging worst case gain upper and lower bound algorithms $U_\epsilon(\cdot)$ and $L_\epsilon(\cdot)$. For given problem instances, obtaining arbitrarily tight bounds may require large computational efforts. In this section, we show that if only suboptimal filters are required, then less inefficient, non-converging worst-case gain algorithms can be used in the design. Roughly speaking, for a given order K, the filter can only be as suboptimal as the gap in the worst-case analysis.

Suppose $U_{sub}(\cdot)$ and $L_{sub}(\cdot)$ are worst case gain upper and lower bound algorithms, respectively, satisfying $U_{sub}(\cdot) - L_{sub}(\cdot) \leq \tau$ on all problem instances, with τ some fixed number. Let

$$\begin{aligned} \Gamma_{sub} &:= L_{sub}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) - \Lambda_{K,\mathcal{F}} \\ \Psi_{sub} &:= L_{sub}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) - \eta_{\mathcal{F}}. \end{aligned}$$

Lemma 21 *Given $\epsilon > 0$, and worst case gain upper and lower bound algorithms U_{sub} and L_{sub}, where L_{sub} also yields a perturbation that achieves the lower bound. If we use Γ_{sub} to substitute Γ in Steps 1 and 2 of Algorithm 1, then for any given filter order K, the algorithm yields a $(\tau + \epsilon)$-suboptimal solution of K-th order filter.*

Proof. Notice that Γ_{sub} can be made arbitrarily small by repeating Steps 1-2 of Algorithm 1. The arguments are the same as in the proof of Lemma 19. So we have

$$\begin{aligned} \max_{\Delta \in \mathbf{B_\Delta}} \|R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta\|_\infty - \Lambda_{K,\mathcal{F}} &\leq U_{sub}\left(R_\Delta + U_\Delta F^*_{K,\mathcal{F}} V_\Delta, \mathbf{\Delta}\right) - \Gamma_{sub} \\ &\leq \tau + \epsilon \end{aligned}$$

in finite iterations. This completes the proof. ■

If we substitute both Ψ_{sub} and Γ_{sub} in Algorithm 1, then other properties are also the same except that we have $\tau + \epsilon$ sub-optimality rather than ϵ sub-optimality.

4.7 Example

The example here is a linear system in state space form with real parameter uncertainties. To transform this back to the form of (1–4) is an easy exercise. Software tools used to solve the optimization problem (in LMIs) are Yalmip (parser) [28] and SeDuMi (solver) [50] running in Matlab. Consider discrete-time system:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.7 & 0.5+0.5\delta \\ -0.5 & 0.6 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(k) \\ y(k) &= \begin{bmatrix} 1 & 0.4 \end{bmatrix} x(k) + 0.2\, d(k) \\ z(k) &= x(k) \end{aligned}$$

where $\delta \in [-1, 1]$.

To illustrate the effects of FIR order, we specify the length of the FIR filter in this example, and only Step 1)-3) of Algorithm 1 are used. We use stop tolerance $\Gamma \leq 0.02$, and start with $\mathcal{F_{\boldsymbol{\Delta}}}^0 = \{0\}$. When $K = 2$, the final uncertainty set is $\mathcal{F}_{\boldsymbol{\Delta}} = \{0, 1, -1\}$ and the worst case performance is 2.724. The two taps are

$$Q_1 = \begin{bmatrix} 0.6368 \\ 0.9025 \end{bmatrix}, \; Q_2 = \begin{bmatrix} 0.3477 \\ -0.8640 \end{bmatrix}.$$

More details are given in the following table:

	$\Lambda_{2,\mathcal{F}} \leq \min_{Q_2} \max_{\Delta \in \mathbf{B_\Delta}} \|\|T_{d \to e}\|\|_\infty \leq \gamma_{2,\mathcal{F}}$	
$\mathcal{F}_{\boldsymbol{\Delta}}$	$\Lambda_{2,\mathcal{F}}$	$\gamma_{2,\mathcal{F}}$
$\{0\}$	0.9054	8.00
$\{0, 1\}$	1.7212	2.841
$\{0, 1, -1\}$	2.7233	2.724

When $K = 5$:

	$\Lambda_{5,\mathcal{F}} \leq \min_{\mathcal{Q}_5} \max_{\Delta \in \mathbf{B_\Delta}} \|\|T_{d\to e}\|\|_\infty \leq \gamma_{5,\mathcal{F}}$	
$\mathcal{F}_{\mathbf{\Delta}}$	$\Lambda_{5,\mathcal{F}}$	$\gamma_{5,\mathcal{F}}$
$\{0\}$	0.8330	7.552
$\{0, 1\}$	1.4394	3.071
$\{0, 1, -1\}$	1.7311	1.733

When $K = 25$:

	$\Lambda_{25,\mathcal{F}} \leq \min_{\mathcal{Q}_{25}} \max_{\Delta \in \mathbf{B_\Delta}} \|\|T_{d\to e}\|\|_\infty \leq \gamma_{25,\mathcal{F}}$	
$\mathcal{F}_{\mathbf{\Delta}}$	$\Lambda_{25,\mathcal{F}}$	$\gamma_{25,\mathcal{F}}$
$\{0\}$	0.8330	7.540
$\{0, 1\}$	1.3189	2.909
$\{0, 1, -1\}$	1.5972	1.711
$\{0, 1, -1, -0.50\}$	1.6081	1.656
$\{0, 1, -1, -0.50, 0.44\}$	1.6109	1.622

Table 4.1 summarizes the performance of FIR filters with different orders. The first column shows the lower bound η of Λ, which is calculated with corresponding $\mathcal{F}_{\mathbf{\Delta}}$ and a uniform frequency grid with 50 points. When the filter order $K = 25$, $\Phi = \gamma - \eta = 0.021$.

Table 4.1: Robust FIR Filters with Different Order

	Upper		
η	$\mathcal{F}_{\mathbf{\Delta}}$	K	$\gamma_{K,\mathcal{F}}$
1.585	$\{0, 1, -1\}$	2	2.724
1.585	$\{0, 1, -1\}$	5	1.733
1.601	$\{0, 1, -1, -0.5, 0.44\}$	25	1.622
Lower			

Table 4.2 compares nominal and worst case performance of the 25th order FIR filter in this chapter with other filter design techniques. The other methods are: design for the nominal model ($\delta = 0$) minimizing the H_∞ norm from $d \to e$, [3]; formulate the worst-case gain minimization as a mixed-μ synthesis, and use the $(D, G) - K$

iteration of [60]; design using LMI techniques from Chapter 3; and, design using LMI techniques, [18], which minimize a bound on the $l_2 \to l_2$ gain from d to e, in the presence of time-varying uncertainties.

Table 4.2: Comparison of Various Filters

	FIR25	Nominal ($\delta = 0$)	μ syn. [60]	Chap. 3	Filter in [18]
Nominal	1.61	**0.833**	1.89	3.10	5.60
Worst case	**1.62**	7.88	2.91	4.02	6.70

The worst case performance in this table are calculated directly over a fine grid of the uncertainty set. Obviously, and as expected, the 25th order FIR filter outperforms all other filters in Table 4.2 in terms of worst case performance.

Fig. 4.4 compares the performance of different filters further. It shows the H_∞ norm from disturbance to estimation error as a function of δ. The lowest curve shows the performance of optimal point-wise H_∞ filters, which are designed at each fixed value of $\delta \in [-1, 1]$. This curve is a lower bound for all filters. Over $-1 \leq \delta \leq 1$, the robust FIR filter proposed in this chapter achieves the best worst case performance. This robust FIR filter also achieves performance better than the robust filter from [18] and the one given in Chapter 3, at each fixed value of δ.

4.8 Conclusions

In this chapter, we show that the robust linear filter design can be cast as a convex optimization problem. The use of finite dimensional approximations is justified and

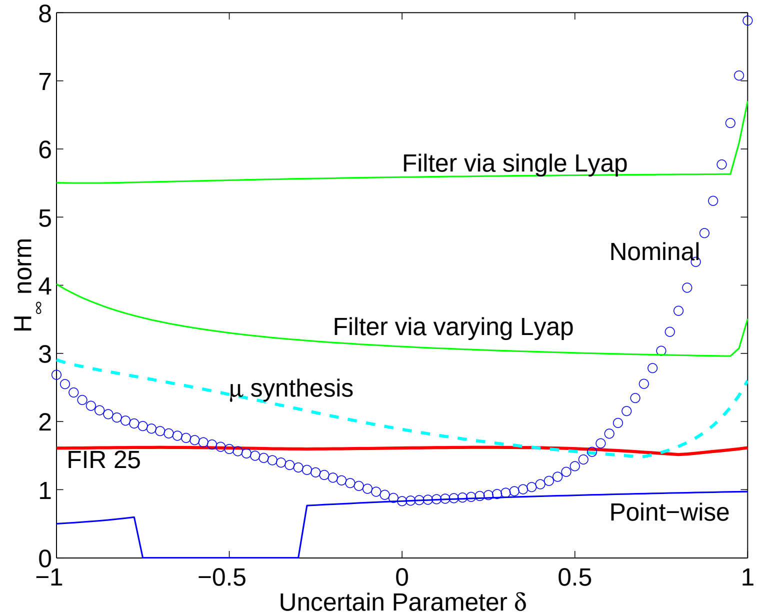

Figure 4.4: Comparison of Various Filters

an algorithm is proposed to carry out the design. In the limit, the result approaches to the optimal solution. An example shows the effectiveness of the proposed algorithm. Future work will address more about the set refinement strategy, which will dramatically affect the computational complexity.

Chapter 5

Controller Synthesis with Input Saturation – Applications of SOS Programming

Control problems for systems subject to input saturation have been studied considerably, and many approaches have been proposed. Anti-windup scheme [51] designs nonlinear compensator to avoid a linear controller saturating, where the linear controller is designed without considering input saturation. Model predictive control, such as [48], also called receding horizon control, calculates constrained control input via on-line optimization. There are also methods based on absolute stability, such as [22], and methods based on constructing control Lyapunov functions (CLF), such as [24], to avoid input saturation. In this chapter, we take the last approach.

In [24], Hu and Lin design a state-feedback controller by constructing a CLF, and this method pertains to linear systems with saturation and linear controllers. The design task results in a semidefinite programming (SDP) problem, which is convex and can be solved efficiently. This chapter is an extension of [24], with the development of a computational framework called sum of squares (SOS) programming [39], which enables the use of semidefinite programming (SDP) to solve optimization problems with polynomial equations and inequalities. Hence problems considered in this chapter are extended to systems with polynomial vector fields and polynomial controllers. The optimization problems formulated are not convex, and heuristic algorithms are proposed. For linear systems with saturation, algorithms in this chapter can be used to improve available results, e.g., those in [24]. Another goal of this chapter is to see how well the SOS programming works in controller synthesis, since for linear system with saturation, we have existing methods to compare to. This chapter is a continue work of [25].

The remainder of this chapter is organized as follows. Section 5.1 introduces the background of SOS programs. In Section 5.2, we design a controller to enlarge a domain of attraction (DOA) for systems with actuator saturation. In this section, an example is provided to be compared with the results in [24]. In Section 5.3, we consider disturbance rejection problems for systems with bounded noises, and examples are also provided. Conclusions are drawn in Section 5.4.

5.1 Preliminaries – P-satz and SOS programs

5.1.1 Positivstellensatz and $\mathcal{S}$-Procedure

A *Monomial* m_α in n variables is a function defined as $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}_+^n$. The degree of a monomial is defined, $\deg m_\alpha := \sum_{i=1}^n \alpha_i$. A *Polynomial* f in n variables is a finite linear combination of monomials, $f := \sum_\alpha c_\alpha m_\alpha = \sum_\alpha c_\alpha x^\alpha$ with $c_\alpha \in \mathbb{R}$. $\mathcal{R}_n$ is the set of all polynomials in n variables. The degree of f is $\deg f := \max_\alpha \deg m_\alpha$ (provided the associated c_α is non-zero). Σ_n is the set of sum of squares (SOS) polynomials in n variables.

$$\Sigma_n := \left\{ p \in \mathcal{R}_n \,|\, p = \sum_{i=1}^t f_i^2 \quad f_i \in \mathcal{R}_n,\, i = 1, \ldots, t \right\}$$

If $p \in \Sigma_n$, then $p(x) \geq 0 \ \forall x \in \mathbb{R}^n$.

Given $\{g_1, \ldots, g_t\} \in \mathcal{R}_n$, the *Multiplicative Monoid* generated by g_j's is the set of all finite products of g_j's, including 1 (i.e. the empty product). It is denoted as $\mathcal{M}(g_1, \ldots, g_t)$. Given $\{f_1, \ldots, f_r\} \in \mathcal{R}_n$, the *Cone* generated by f_i's is

$$\mathcal{P}(f_1, \ldots, f_r) := \left\{ s_0 + \sum_{i=1}^l s_i b_i \,|\, l \in \mathbb{Z}_+, s_i \in \Sigma_n, b_i \in \mathcal{M}(f_1, \ldots, f_r) \right\}.$$

Note that if $s \in \Sigma_n$ and $f \in \mathcal{R}_n$, then $f^2 s \in \Sigma_n$ as well. This allows us to express a cone of $\{f_1, \ldots, f_r\}$ as a sum of 2^r terms.

Given $\{h_1, \ldots, h_u\} \in \mathcal{R}_n$, the *Ideal* generated by h_k's is

$$\mathcal{I}(h_1, \ldots, h_u) := \left\{ \sum h_k p_k \,|\, p_k \in \mathcal{R}_n \right\}.$$

With these definitions we can state the following theorem from [4, Theorem 4.2.2].

Theorem 14 (Positivstellensatz) *Given polynomials* $\{f_1, \ldots, f_r\}$, $\{g_1, \ldots, g_t\}$, *and* $\{h_1, \ldots, h_u\}$ *in* $\mathcal{R}_n$. *The set*

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} f_1(x) \geq 0, \ldots, f_r(x) \geq 0 \\ g_1(x) \neq 0, \ldots, g_t(x) \neq 0 \\ h_1(x) = 0, \ldots, h_u(x) = 0 \end{array} \right\}$$

is empty if and only if there exist polynomials $f \in \mathcal{P}(f_1, \ldots, f_r)$, $g \in \mathcal{M}(g_1, \ldots, g_t)$, $h \in \mathcal{I}(h_1, \ldots, h_u)$ *such that* $f + g^2 + h = 0$.

The following simple lemma is a generalization of the well known S-procedure [6], and it is a special case of P-satz.

Lemma 22 (Generalized S-procedure) *Given* $\{p_i\}_{i=0}^m \in \mathcal{R}_n$. *If there exist* $\{s_k\}_{i=1}^m \in \Sigma_n$ *such that* $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, *then*

$$\bigcap_{i=1}^m \{x \in \mathbb{R}^n : p_i(x) \geq 0\} \subset \{x \in \mathbb{R}^n : p_0(x) \geq 0\}.$$

Proof. Since $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, so $p_0 \geq \sum_{i=1}^m s_i p_i \; \forall x$. For any $x \in \bigcap_{i=1}^m \{x \in \mathbb{R}^n : p_i(x) \geq 0\}$, since $s_i(x) \geq 0$, so $\sum_{i=1}^m s_i p_i \geq 0$, hence $p_0(x) \geq 0$. This completes the proof. ∎

5.1.2 SOS Programming

Many problems involving polynomials can be transformed to check whether some polynomials are SOS polynomials or to search over convex sets for SOS polynomials.

We call these problems as SOS programs. Parrilo [39] showed that these problems can be posed as SDP problems.

Theorem 15 (Parrilo) *Given a finite set* $\{p_i\}_{i=0}^m \in \mathcal{R}_n$*, and convex set* $\mathcal{C} \subset \mathbb{R}^m$ *defined by semidefinite constraints. The existence of* $\lambda \in \mathcal{C}$ *such that*

$$p_0 + \sum_{i=1}^m \lambda_i p_i \in \Sigma_n$$

is a SDP feasibility problem.

This theorem is important since it provide basic computation tools in SOS programming. A formal statement of this is given in [39], and here we present a simplified version:

Corollary 2 *Given* $p_0, p_1 \in \mathcal{R}_n$*, searching* $k \in \mathcal{R}_n$ *of a given degree, such that* $p_0 + kp_1 \in \Sigma_n$ *is a SDP problem.*

Proof. Write k as a linear combination of its monomials $\{m_j\}$, $k = \sum_{j=1}^s a_j m_j$. Rewrite $p_0 + kp_1$ using this decomposition $p_0 + kp_1 = p_0 + \sum_{j=1}^s a_j(m_j p_1)$, which since $(m_j p_1) \in \mathcal{R}_n$ is a feasibility problem that can be checked by Theorem 15. ■

Software packages, such as, SOSTOOLS [45] and GLOPTIPOLY [21], exist to formulate SOS programs as SDPs, and these packages use Sturm's semidefinite programming solver SeDuMi [50]. Examples in this chapter are implemented using SOSTOOLS.

5.2 State-feedback to enlarge a domain of attraction

Consider a time-invariant nonlinear system which is affine in controls:

$$\dot{x} = f(x) + g(x)\,\mathrm{sat}(u), \tag{5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and f, g are n-vectors of elements of $\mathcal{R}_n$ such that $f(0) = 0$. Function $\mathrm{sat}(\cdot)$ is the standard saturation function. We want to synthesize a state feedback controller $u = K(x)$ to enlarge the set of points, with a given shape, that are attracted to the origin:

Problem 1 *Define the region interested in expanding as $P_\beta := \{x \in \mathbb{R}^n | p_x(x) \le \beta\}$, for some given convex function $p_x \in \Sigma_n$. Find a state feedback controller $K(x)$ to maximize β, such that $\mathcal{I} \supset P_\beta$, and $\mathcal{I}$ is a domain of attraction.*

A sufficient condition for a set to be invariant is derived using a control Lyapunov function:

Theorem 16 *Given system (5.1), if there exist $K(x)$ and $V(x)$ satisfying*

$$V(x) > 0 \;\forall\; x \in \mathbb{R}^n \neq 0,\;\; V(0) = 0 \tag{5.2}$$

$$\{x \in \mathbb{R}^n | V(x) \le 1\} \setminus \{x = 0\} \subset \left\{ x \in \mathbb{R}^n \left| \frac{\partial V}{\partial x}(f(x) + g(x)K(x)) < 0 \right. \right\} \tag{5.3}$$

$$\{x \in \mathbb{R}^n | V(x) \le 1\} \subset \{x \in \mathbb{R}^n | K(x) \le 1\} \tag{5.4}$$

$$\{x \in \mathbb{R}^n | V(x) \le 1\} \subset \{x \in \mathbb{R}^n | K(x) \ge -1\}, \tag{5.5}$$

then $V(x)$ is a CLF, and with controller $K(x)$, every trajectory started in $\mathcal{I} := \{x \in \mathbb{R}^n | V(x) \leq 1\}$ converges to the origin.

Proof. On set $\{x|V(x) < 1\}\backslash\{0\}$, condition (5.2) ensures that $V(x)$ is positive, while (5.3) says $\dot{V}(x) < 0$. So $V(x)$ is a Lyapunov function for the closed-loop system. Condition (5.4) and (5.5) are the constraints for input saturation. By Lyapunov theorem, we know that when above conditions are met, the level set $\{x|V(x) \leq 1\}$ is a domain of attraction. ■

The use of number "one" in (5.3) defining $\mathcal{I}$ does not loss any generality, since we can always scale $V(x)$. Based on Theorem 16, Problem 1 can be reformulated as

$$\max \quad \beta \qquad \text{over } K, V \tag{5.6}$$

$$\text{s.t.} \quad (5.2 - 5.5) \tag{5.7}$$

$$\{x \in \mathbb{R}^n | p_x(x) \leq \beta\} \subset \{x \in \mathbb{R}^n | V(x) \leq 1\} \tag{5.8}$$

Note that in above problem, K and V can be any function of x. The difficulty is how to search K and V over functions of x, satisfying the set containment conditions. For linear systems with saturation and using quadratic Lyapunov functions and linear controllers, this problem can be formulated as SDPs, see [24]. For systems with polynomial vector fields and feedback control laws, we can use SOS programs to carry out the design.

5.2.1 Controller design via SOS programming

The goal now is to solve (5.6-5.8) with SOS programming. Let $V(x)$ be a SOS polynomial without constant term, i.e., $V(0) = 0$, and be $K(x)$ a rational function, with a given positive definite polynomial $d(x)$ as denominator, and write $K(x) = \frac{K_n(x)}{d(x)}$, where $K_n(x)$ is a polynomial to be determined.

First, need to change (5.6-5.8) to problems that involve checking SOS polynomials. Let $l_1(x)$ and $l_2(x)$ be any polynomials that $l_i(0) = 0$ and $l_i(x) > 0$ as $x \neq 0$, $i = 1, 2$, and we have

$$\max \quad \beta \qquad \text{over } K_n \in \mathcal{R}_n \quad V, s_6, s_8, s_{10}, s_{11} \in \Sigma_n \quad \beta, \epsilon_1, \epsilon_2 \in \mathbb{R}^+ \tag{5.9}$$

$$\text{s.t.} \quad V - \epsilon_1 l_1 \in \Sigma_n \tag{5.10}$$

$$-\Big((\beta - p_x) + (V-1)s_6\Big) \in \Sigma_n \tag{5.11}$$

$$-\left((1-V)s_8 + \frac{\partial V}{\partial x}(df + gK_n) + \epsilon_2 l_2\right) \in \Sigma_n \tag{5.12}$$

$$\Big((d - K_n) - (1-V)s_{10}\Big) \in \Sigma_n \tag{5.13}$$

$$\Big((d + K_n) - (1-V)s_{11}\Big) \in \Sigma_n \tag{5.14}$$

Conditions (5.10-5.14) are sufficient for (5.7-5.8). Indeed, obviously (5.10) is sufficient for (5.2). By Lemma 22, (5.11) and (5.13-5.14) are sufficient for (5.8) and (5.4-5.5), respectively. Using Lemma 22, it is easy to show that (5.12) is sufficient for (5.3).

In above equations, the degree of each polynomial variable is not given explicitly. They are free parameter to be determined. For polynomial p, we use n_p to denote its degree. Inequalities (5.10 - 5.14) add the following constraints on the degree of polynomial variables implicitly: $n_V \geq n_l,\ n_{p_x} \geq n_V + n_{s6},\ n_{s8} \geq \max\{n_g + n_{K_n}, n_f + n_d, n_{l_2} - n_V + 1\} - 1,\ n_{s10} + n_V \geq n_{K_n},\ n_{s11} + n_V \geq n_{K_n}$. In general, these degrees are specified before design.

Unfortunately, SOS programming can not be used to solve (5.9-5.14) directly, because the problem is not jointly convex in all the polynomial variables. We propose a heuristic algorithm to solve it, which requires non-trivial initial conditions to start with. In the following, suppose linearizations of given systems are stabilizable, so that linear design methods work locally. Input saturation is not an obstacle, since there is always a region small enough that the controller does not saturate.

Algorithm 2 (Controller Design Algorithm) *Specify the maximum degrees of* K_n, V *and* s_i, $i = 2, 8, 10, 11$. *Set* l_1 *and* $l_2 = \sum x_i^{n_V}$.

1. *Find an initial control Lyapunov function* V_0, *by solving any Riccati equation of the linearized system at the origin.*

2. *Maximize* β *over* $K_n, s_6, s_8, s_{10}, s_{11}, \epsilon_2$. *This has to be done in two steps, because* $K_n(x)$ *does not affect* β *directly.*

   (a) *Maximize* α *over* $K_n, s_8, s_{10}, s_{11}, \epsilon_2$, *while keep* V *fixed, such that*

$$-\left((\alpha - V)s_8 + \frac{\partial V}{\partial x}(fd + gK_n) + \epsilon_2 l_2\right) \in \Sigma_n$$

$$\Big((d - K_n) - (\alpha - V)s_{10}\Big) \in \Sigma_n$$

$$\Big((d + K_n) - (\alpha - V)s_{11}\Big) \in \Sigma_n.$$

For each fixed α, this is a convex feasibility problem. Additionally, if for any α_0 that this problem is feasible, then so does for each $0 < \alpha < \alpha_0$. So we can use line-search over α. Then set $V = V/\alpha$, $s_{10} = \alpha s_{10}$, $s_{11} = \alpha s_{11}$ and $\epsilon_2 = \epsilon_2/\alpha$.

(b) *Maximize β over s_6, such that*

$$-\Big((\beta - p) + (V - 1)s_6\Big) \in \Sigma_n.$$

3. *Maximize β over V (with $s_6, s_8, s_{10}, s_{11}, \epsilon_1$ and K_n, ϵ_2 obtained from Step 2)), such that (5.10 - 5.14) are satisfied. This is a convex problem.*

4. *Repeat 2-3 until β stop improving.*

This V-K iteration may converge to local maxima.

5.2.2 Comparisons and a numerical example

Theorem 2 of [23] shows that for linear systems, if we use quadratic Lyapunov functions, linear controllers are enough.

Theorem 17 ([23]) *Given an ellipsoid $\mathcal{E}(P, \rho)$ and an $F \in \mathbb{R}^{1\times n}$, suppose that $(A + BF)^T P + P(A + BF) < 0$. Then $\mathcal{E}(P, \rho)$ is contractive invariant under $u = \text{sat}(Fx)$ if and only if there exists a function $h(x) : \mathbb{R}^n \to \mathbb{R}$, $|h(x)| < 1$ for all $x \in \mathcal{E}(P, \rho)$ is contractive invariant under the control $u = h(x)$, i.e., $x^T P(Ax + Bh(x)) < 0 \ \forall \ x \in \mathcal{E}(P, \rho)\backslash\{0\}$.*

Moreover, in [24] the problem is transformed into a convex one, and the global maximum can be achieved. Hence, in this case, Algorithm 2 can not obtain better results compared to those in [24]. In order to get a larger domain of attraction, we must use higher degree Lyapunov functions V, as an example shows later.

This numerical example is drawn from [24] for comparison purposes. It is an anti-stable linear system with actuator saturation. The system is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{sat}(u).$$

The algorithm proposed in [24] yields control law $F_1^* = [-0.0025 \; -0.2987]$, and domain of attraction $\mathcal{E}(P_1^*, 1) := \{x : x^T P_1^* x < 1\}$, where

$$P_1^* = \begin{bmatrix} 0.0752 & -0.0566 \\ -0.0566 & 0.1331 \end{bmatrix}.$$

Using the algorithm proposed in this chapter, with $p = x^T P_1^* x + 10^{-6}(x_1^6 + x_2^6) + 10^{-6}(x_1^{10} + x_2^{10})$ and $d = 1$, we carried out several designs with different degrees of polynomial variables and a randomly picked initial CLF. [1] Table 5.1 summarizes these results.

It shows that when V is quadratic, results here is worse than the result in [24], though when $n_K = 5$, we got almost the same result. As the degree of $V(x)$ gets higher, larger domain of attraction are obtained. The maximal ellipsoid we got is $\mathcal{E}(P_1^*, 1.19)$, which is better than the result in [24]. Figure 5.1 shows the domain

[1] We use this p rather than $x^T P_1^* x$ because of the degree constraints. An alternative is to change (5.11) to $-\Big((\beta - p)s_6 + (V - 1)\Big) \in \Sigma_n$, but then have to use line-search over β in this sub-step.

Table 5.1: DOA with different parameters

β	n_V	n_K	n_{s6}	n_{s8}	n_{s10}, n_{s11}
0.6257	2	1	4	4	4
0.9982	2	5	4	4	4
1.050	4	5	4	4	4
1.194	6	5	4	4	4

of attraction got in this chapter, together with the ellipse inside and the ellipse in chapter [24].

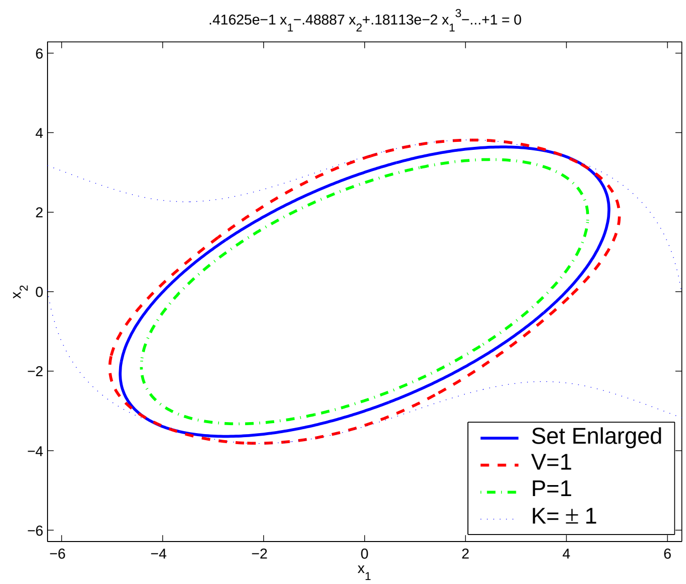

Figure 5.1: saturation and level sets

From the figure, we can see that though S-procedure is only a sufficient condition, it works very well in this example.

Several other randomly picked initial CLFs were used, and similar results were obtained. When the initial CLF is the result from [24], we obtained similar (the difference is due to numerical problems) result in the case of quadratic CLFs, and better results when higher order CLFs are used. So Algorithm 2 can also be used to improve the results in [24].

5.3 State-feedback for disturbance rejection

In this section, we consider system (5.1) with an additional noise term with bounded magnitude:

$$\dot{x}(t) \quad = \quad f(x) + g_u(x)\,\mathrm{sat}(u(t)) + g_w(x)w, \tag{5.15}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and f, g_u and $g_w(x)$ are n-vectors of elements of $\mathcal{R}_n$ such that $f(0) = 0$. Disturbance $w(t) \in \mathbb{R}$ belongs to set $\mathbf{B}_w := \{w : w(t)^2 \leq 1, \ \forall t \geq 0\}$.

Two objectives are considered in this section. The first one is to enlarge an invariant set in the presence of disturbance w, and the second one is to reject disturbance, i.e., to have a small invariant set containing the origin so that a trajectory starting from the origin will stay close to the origin. For system (5.15), given functions p_{X_0} and p_{X_∞}, the following problems are considered:

Problem 2 (Invariant set enlargement) *Design $K(\cdot)$ such that the closed-loop system has an invariant set $\mathcal{I} \supset \{x : p_{X_0} \leq \alpha_2\}$ with α_2 maximized.*

Problem 3 (Disturbance rejection) *Design $K(\cdot)$ such that the closed-loop system has an invariant set $\mathcal{I} \subset \{x : p_{X_\infty} < \alpha_3\}$ with α_3 minimized.*

Problem 4 (Disturbance rejection with guaranteed domain of attraction) *Design $K(\cdot)$ such that the closed-loop system has an invariant set $\mathcal{I} \supset \{x : p_{X_0} \leq 1\}$, and for all x_0 in this set, the trajectory will enter a smaller invariant set $\mathcal{I} \subset \{x : p_{X_\infty} < \alpha_4\}$ with α_4 minimized.*

Problem 4 is the main problem we want to solve, and the results of Problem 2 and 3 serve as initial conditions. Let $V(x)$ be a polynomial without constant term, and $K(x) = \frac{K_n(x)}{d(x)}$, where $K_n(x)$ is a polynomial to be designed and $d(x)$ is a given positive definite (PD) polynomial.

5.3.1 Invariant set enlargement

Using Lyapunov arguments, we can derive a sufficient condition for a set $\mathcal{I}$ to be invariant:

Lemma 23 *Given nonlinear system (5.15) and PD polynomial $d(x)$, if there exist polynomials $K_n(x)$, $V(x)$, p_k, s_2, s_{10}, s_{11} and ϵ_1 satisfying (5.10), (5.13), (5.14) and*

$$\begin{aligned}&(1 - V(x))p_k(x) - (1 - w^2)s_2(x)\\ &\quad -\frac{\partial V}{\partial x}\Big(f(x)d(x) + g_u(x)K_n(x) + g_w(x)d(x)w\Big) \in \Sigma_{n+1}\end{aligned} \tag{5.16}$$

then with controller $K(x) = \frac{K_n(x)}{d(x)}$, $\mathcal{I} := \{x : V(x) \leq 1\}$ is an invariant set in the presence of disturbance w.

Proof. By Lyapunov arguments, the following are sufficient conditions for $\mathcal{I}$ to be invariant:

$$(5.2), (5.4) \text{ and } (5.5) \tag{5.17}$$

$$\{(x,w):\, V(x)=1,\; w^2\le 1\}\subset\{x:\, \dot V(x)<0\}. \tag{5.18}$$

We know for sets A and B, $A\subset B \Leftrightarrow A\cap\overline{B}=\emptyset$. So (5.18) can be rewritten as:

$$\{(x,w):\, V(x)=1\}\cap\{(x,w):\, 1-w^2\ge 0\}\cap\{(x,w):\, \dot V(x)\ge 0\}=\emptyset. \tag{5.19}$$

Since $d(x)>0$ for all x, we know $\{(x,w):\, \dot V(x)\ge 0\}$ equivalent to:

$$\left\{(x,w):\, \frac{\partial V}{\partial x}\Big(f(x)d(x)+g_u(x)K_n(x)+g_w(x)d(x)w\Big)\ge 0\right\}.$$

Invoking P-satz, we have the following sufficient condition for (5.19): $\exists p_k\in\mathcal{R}_{n+1}$ and $s_1, s_2\in\Sigma_{n+1}$ such that

$$(1-V(x))p_k-(1-w^2)s_2-\frac{\partial V}{\partial x}\Big(fd+g_uK_n+g_wdw\Big)s_1\in\Sigma_{n+1}.$$

Since $s_1\in\Sigma_{n+1}$, for simplicity, let $s_1=1$, which leads to a sufficient condition (5.16). As before, sufficient conditions for (5.2), (5.4) and (5.5) are (5.10), (5.13) and (5.14), respectively. Thus the proof is complete. ∎

Note that we only need to guarantee inputs not saturating on the boundary of $\{V=1\}$, so we should use $\{|K|\le 1\}\supset\{V=1\}$ rather than (5.4-5.5), i.e., $\{|K|\le 1\}\supset\{V\le 1\}$. We made this choice because this result is used for Problem 4 later. Problem 2 can be reformulated as follows, recall $\{x:\, p_{X_0}\le\alpha_2\}\subset\{x: V(x)\le 1\}$:

$$\max \quad \alpha_2 \quad \text{over } \alpha_2, V, K, p_k, s_2, s_6, s_{10}, s_{11}, \epsilon_1 \tag{5.20}$$

$$\text{s.t.} \quad (5.10), (5.13) \text{ and } (5.14) \tag{5.21}$$

$$(1 - V) - (\alpha_2 - p_{X_0})s_6 \in \Sigma_n \tag{5.22}$$

$$(1 - V)p_k - (1 - w^2)s_2 - \frac{\partial V}{\partial x}\Big(fd + g_u K_n + g_w dw\Big) \in \Sigma_{n+1} \tag{5.23}$$

This formulation is not convex in all the variables, and we use ad-hoc "V-K" iteration to solve the problem. The algorithm might converge to local maxima, and non-trivial initial conditions are required. We can use existing techniques to obtain initial CLF $V_0(x)$, such as [24] or the result of Problem 1 in Section 5.2. The constraints on the degrees of polynomial variables are: $n_V \geq n_l,\ n_{p_{x0}} + n_{s_6} \geq n_V,\ n_{p_k} \geq \max(n_f + n_d, n_{g_u} + n_{K_n}, n_{g_w} + n_d) - 1,\ n_{s10} + n_V \geq n_K,\ n_{s11} + n_V \geq n_K$.

Algorithm 3 *(Invariant set enlargement in the presence of disturbance)*

1. *Controller K iteration (V is fixed)*

   (a) *Design K_n*

$$\max \quad \alpha \quad \text{over } K_n \in \mathcal{R}_n,\ p_k \in \mathcal{R}_{n+1},\ s_{10}, s_{11} \in \Sigma_n,$$
$$s_2 \in \Sigma_{n+1},\ \text{and } \alpha > 0$$

$$\text{s.t.} \quad (d - K_n) - (\alpha - V)s_{10} \in \Sigma_n$$

$$(d + K_n) - (\alpha - V)s_{11} \in \Sigma_n$$

$$(\alpha - V)p_k - (1 - w^2)s_2 - \frac{\partial V}{\partial x}\Big(fd + g_u K_n + g_w dw\Big) \in \Sigma_{n+1}$$

   Non-convex, line-search over $\alpha \geq 1$.

   Set $V = V/\alpha$, $s_2 = s_2/\alpha$, $s_{10} = \alpha s_{10}$ and $s_{11} = \alpha s_{11}$.

(b) Invariant set $\mathcal{I}$ update

$$\begin{aligned} \max \quad & \alpha_2 \quad \text{over } s_6 \in \Sigma_n,\ \alpha_2 \in \mathbb{R}^+ \\ \text{s.t.} \quad & (1 - V) - (\alpha_2 - p_{X_0}) s_6 \in \Sigma_n \end{aligned}$$

Non-convex, line-search over α_2.

2. *Lyapunov function V iteration ($K_n, p_k, s10, s11$ are fixed.)*

$$\begin{aligned} \max \quad & \alpha_2 \quad \text{over} \quad V, s_6 \in \Sigma_n,\ \ s_2 \in \Sigma_{n+1}\ \epsilon_1 \in \mathbb{R}^+ \\ \text{s.t.} \quad & (5.21) - (5.23) \end{aligned}$$

Non-convex, line-search over α_2 ($\geq \alpha_2$ obtained in Step 1.b)).

3. *The algorithm terminates when the improvement of α_2 is less than a specified value.*

The design algorithm for Problem 3 is similar and we only give the reformulated optimization problem. Recall we want $\{x : p_{X_\infty} \leq \alpha_3\} \supseteq \{x : V(x) \leq 1\}$:

$$\begin{aligned} \min \quad & \alpha_3 \quad \text{over } \alpha_3, V, K, p_k, s_2, s_6, s_{10}, s_{11}, \epsilon_1 \\ \text{s.t.} \quad & (5.10), (5.13) \text{ and } (5.14) \\ & (\alpha_3 - p_{X_\infty}) s_6 - (1 - V(x)) \in \Sigma_n \\ & (1 - V(x)) p_k - (1 - w^2) s_2 - \frac{\partial V}{\partial x}\Big(fd + g_u K_n + g_w dw\Big) \in \Sigma_{n+1} \end{aligned}$$

5.3.2 Controller design for Problem 4

Problem 4 is a combination of Problem 2 and 3, which contains possibly conflicting objectives. So we first determine an invariant set by solving Problem 2, then shrink this region, and design a controller to achieve better disturbance rejection on this region.

Theorem 18 *Given system (5.15), polynomials p_{X_0}, p_{X_∞} convex, and $d(x)$ positive definite. Optimization problem (5.24-5.28) generates control law $K(x) = \frac{K_n(x)}{d(x)}$ and CLF $V(x)$, such that the closed loop system has an invariant set $\mathcal{I} = \{x : V(x) \leq 1\} \supseteq \{x : p_{X_0} \leq 1\}$. Furthermore, for any x starts inside $\mathcal{I}$, the trajectory enters $\{x : p_{X_\infty} \leq \alpha_4\}$ eventually.*

$$\min \quad \alpha_4 \text{ over } K \in \mathcal{R}_n,\ V, s_6, s_{10}, s_{11}, s_{12} \in \Sigma_n,\ s_7, s_8, s_9 \in \Sigma_{n+1},$$

$$\epsilon_1 > 0,\ \alpha_5 \in (0,1) \tag{5.24}$$

$$\text{s.t.} \quad (5.10), (5.13), (5.14) \tag{5.25}$$

$$(1 - V) - (1 - p_{X_0})s_6 \in \Sigma_n \tag{5.26}$$

$$-\frac{\partial V}{\partial x}\Big(fd + g_u K_n + g_w dw\Big) - s_7(1 - V)$$

$$-s_8(1 - w^2) - s_9(V - \alpha_5) \in \Sigma_{n+1} \tag{5.27}$$

$$(\alpha_4 - p_{X_\infty})s_{12} - (\alpha_5 - V) \in \Sigma_n \tag{5.28}$$

Proof. By Lyapunov arguments, for $0 < \alpha_5 < 1$, $\mathcal{I}$ is an invariant set, and $\{x : V(x) < \alpha_5\}$ is an ω-limit set if there exist V and K, such that (5.2), (5.4), (5.5) and the following is satisfied,

$$\{(x,w) : V(x) \leq 1, |w| \leq 1\} \setminus \{x = 0\} \subset \left\{(x,w) : \dot{V}(x) < 0\right\} \cup \{x : V(x) < \alpha_5\} \tag{5.29}$$

Problem 4 can be formulated as follows, where α_4 quantifies the size of the small invariant set:

$$\min \quad \alpha_4 \tag{5.30}$$

$$\text{s.t.} \quad (5.2), (5.4), (5.5) \text{ and } (5.29) \tag{5.31}$$

$$\{x : p_{X_0}(x) \leq 1\} \subset \{x : V(x) \leq 1\} \tag{5.32}$$

$$\{x : V(x) < \alpha_5\} \subset \{p_{X_\infty}(x) \leq \alpha_4\} \tag{5.33}$$

For conditions (5.2), (5.4), (5.5), (5.32) and (5.33), use Lemma 22 as before. Now consider (5.29). Since $V(0) = 0$, so $\{x = 0\} \subset \{x \in \mathbb{R}^n | V(x) \leq \alpha_5\}$, and hence (5.29) is equivalent to

$$\{x \in \mathbb{R}^n,\ w \in \mathbb{R} :\ V(x) \leq 1, |w| \leq 1\} \subset \{x \in \mathbb{R}^n,\ w \in \mathbb{R} :\ V(x) < \alpha_5\} \cup \left\{x \in \mathbb{R}^n,\ w \in \mathbb{R} :\ \frac{\partial V}{\partial x}(f(x)d(x) + g_u(x)K(x) + g_w(x)d(x)w) < 0\right\}.$$

A simple fact for sets A, B and C: $A \cup B \supset C \Leftrightarrow C \cap (\overline{A \cup B}) = \emptyset \Leftrightarrow C \cap \overline{A} \cap \overline{B} = \emptyset$. Condition (5.29) can be written as

$$\left\{ x \in \mathbb{R}^n, w \in \mathbb{R} \;\middle|\; \begin{array}{c} 1 - V(x) \geq 0 \\ 1 - w^2 \geq 0 \\ \frac{\partial V}{\partial x}(f(x)d(x) + g_u(x)K(x) + g_w(x)d(x)w) \geq 0 \\ V(x) - \alpha_5 \geq 0 \end{array} \right\} = \emptyset.$$

Invoking Theorem 22 and making some simplifications, we obtain a sufficient condition (5.27). Put all conditions together, we can get the optimization problem. ■

In the following design algorithm, the result of Problem 2 can be used as initial CLF V_0.

Algorithm 4 *(Controller design for disturbance rejection)*

1. *Controller K iteration (V is fixed)*

   (a) *Design K_n*

$$\begin{aligned} \min \quad & \alpha_5 \quad \text{over } K \in \mathcal{R}_n,\ s_7, s_8, s_9, s_{10} \in \Sigma_{n+1},\ s_{11} \in \Sigma_n,\ \alpha_5 \in (0,1) \\ \text{s.t.} \quad & (5.13),\ (5.14) \text{ and } (5.26) \end{aligned}$$

   Non-convex, using line-search over α_5.

   (b) *Invariant set $\mathcal{I}$ update*

$$\begin{aligned} \min \quad & \alpha_4 \quad \text{over } s_{12} \in \Sigma_n,\ \alpha_4 > 0 \\ \text{s.t.} \quad & (\alpha_4 - p_{X_\infty})s_{12} - (\alpha_5 - V) \in \Sigma_n \end{aligned}$$

   Non-convex, using line-search over α_4.

2. *Lyapunov function V iteration (K, s_7, s_9 s_{10} and s_{11} are fixed)*

$$\begin{aligned} \min \quad & \alpha_4 \qquad \text{over} \quad V, s_6, s_8, s_{12} \in \Sigma_n,\ \epsilon_1, \alpha_4 > 0,\ \alpha_5 \in (0,1) \\ \text{s.t.} \quad & (5.25 - 5.28) \end{aligned}$$

   Non-convex, using line-search over α_4 (less than α_4 obtained in Step 1.b).)

Degree constraints are $n_V \geq n_l$, $n_{p_{x0}} + n_{s6} \geq n_V$, $\max(n_{s7}, n_{s8} - n_V) \geq \max(n_{s9} + 1, n_f + n_d, n_{g_u} + n_{K_n}, n_{g_w} + n_d) - 1$, $n_V \geq n_{p_{x\infty}} + n_{s12}$, $n_{s10} + n_V \geq n_K$, $n_{s11} + n_V \geq n_K$.

5.3.3 Numerical examples

The dynamic system in this section differs from the one in Section 5.2.2 only in the additional noise term:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{sat}(u) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w.$$

In Problem 2, let $p_{X_0} = x_1^2 + x_2^2$ and $d = 1$, and the following degrees of polynomials are used:

polynomial	V	K	s_2	s_6	s_{10}	s_{11}	p_k
variables	x	x	x, w	x	x	x	x, w
degree	4	5	8	4	4	4	4

Because the problem is not convex and ad-hoc algorithm is used, the result depends on the initial condition. With initial CLF V_0 being the 4th degree V in Section 5.2.2, we obtained a ball with radius 2.6020. Some other randomly picked initial CLFs result in balls with radius around 2. In [24], the maximum invariant ball has radius 2.3195. Figure 5.2 shows the invariant set obtained in this chapter, together with the ball inside and the ball obtained in chapter [24].

Now, consider Problem 4. Let $p_{X_0} = \frac{1}{4}(x_1^2 + x_2^2)$ and $p_{X_\infty} = x_1^2 + x_2^2$ for comparison purposes, and $d = 1$. Degrees of polynomials involved in Problem 4 are as follows:

polynomial	K	V	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}
variables	x	x	x	x, w	x, w	x, w	x	x	x
degree	5	4	4	4	4	4	4	4	2

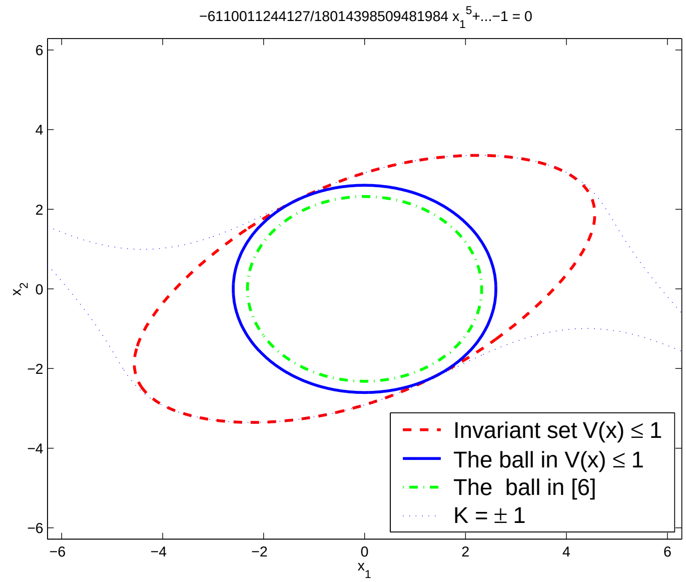

Figure 5.2: The invariant ellipsoids and input saturation

We use the result of Problem 2 as the initial CLF in Algorithm 4, and we obtain $\alpha_4 = 0.110$, which is much better than the result in [24], where $\alpha_4 = 0.9725$. We did simulations for both controllers, with $x(0) = 0$ and $d(t) = \sin(0.02t)$. Fig. 5.3 shows that the controller in this chapter indeed works better, and the actual performance of the controller in [24] is better than what $\alpha_4 = 0.9725$ indicates.

5.4 Conclusions

State feedback controller synthesis problems for systems with polynomial vector fields and input saturations are considered. Ad-hoc algorithms are proposed since these synthesis problems are not convex, and sum of squares (SOS) programming

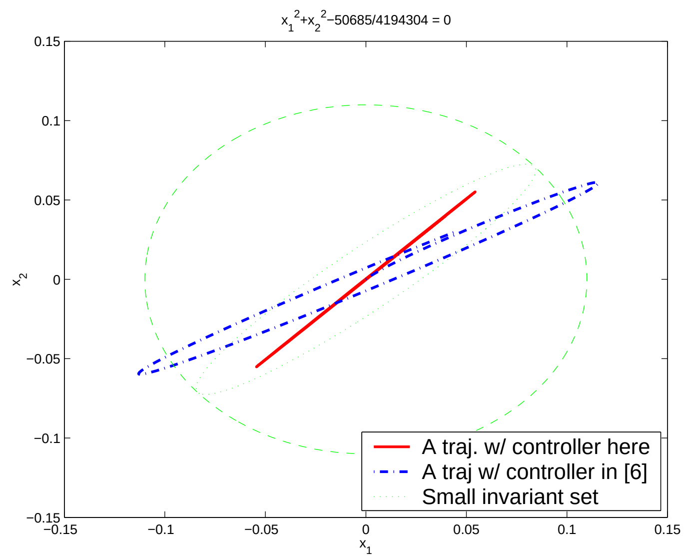

Figure 5.3: Smaller invariant set and simulation results

is the computational tool. Examples show that these algorithms work well. This chapter provides some new techniques for controller synthesis with input saturation, as well as shows that SOS programming is useful in these problems.

Chapter 6

Conclusions

In this dissertation, robust robust H_2 and H_∞ filter design problems for systems described by time-varying LFT uncertain models are considered. The uncertainties can be either unstructured or structured, and they are norm bounded. The main result is that after upper-bounding the objectives, the problems of minimizing the upper bounds are converted to finite dimensional convex optimization problems involving linear matrix inequalities (LMIs), and they can be solved very efficiently.

The contribution is that the treatment of norm bounded (both structured and unstructured) LFT uncertainty using LMI (rather than Riccati) methods. In the norm bounded unstructured uncertainty case, we establish necessary and sufficient LMI conditions for finding the upper bounds, which are less conservative than those methods based on Riccati equations [42, 56]. These are extensions of well known results for systems with polytopic uncertainty.

We also consider robust H_∞ filter design for systems with time-invariant parameter uncertainties in a polytope. Systems considered in this chapter are discrete-time systems. The problem is also transformed to LMIs based another upper bound. This is an extension of the results presented in [19].

A direct method for optimal worst case robust linear filter design is proposed in Chapter 4. This is based on the observation that the design problem, which is infinite dimensional, is convex in the filter. We show that finite dimensional relaxations can be used to get arbitrarily close to the optimal solution. The design procedure constitutes successive finite dimensional approximations, involving worst case analysis to get upper and lower bounds. This approach differs from standard robust filtering techniques. Usually, these minimize a specific choice of upper bound of the objective function. The choice is usually well-motivated, but partially made for computational simplicity. The computational demands put forth here are much larger.

Two branch & bound algorithms for worst case analysis are outlined. One is for theoretical interest and the other one is for practical purpose. These algorithms extend ideas in [33] to matrices with additional complex perturbations (unmodelled dynamics). Another contribution here is that we propose a method to perform this calculation cross frequency.

Polynomial state feedback controller synthesis for systems with polynomial vector fields, which is subject to bounded control input, are considered. Two kinds of problems are considered. The first one is to design a controller to enlarge a domain of attraction (DOA), and the second is about disturbance rejection. Sum of squares (SOS) programming is the computational tool. These synthesis problems are not convex, and ad-hoc algorithms are proposed. We extend the result of [24] to systems with polynomial vector fields and polynomial controllers. For the cases covered by [24], algorithms here can be used to improve available results. This also shows that SOS programming works in controller synthesis.

Bibliography

[1] B. Anderson and J. B. Moore. *Optimal Filtering.* Prentice–Hall, 1979.

[2] V. Balakrishnan, S. Boyd, and S. Balemi. Branch and bound algorithm for computing the minimum stability degree of parameter-dependent linear systems. Technical report, ISL, Stanford Univ, 1991.

[3] G. J. Balas, J. C. Doyle, K. Glover, A. K. Packard, and R. Smith. *μ-Analysis and Synthesis Toolbox: For Use with MATLAB.* The MathWorks, Inc., 3 edition, 2001.

[4] J. Bochnak, M. Coste, and M-F. Roy. *Géométrie algébrique réelle.* Springer, Berlin, 1986.

[5] S. Boyd, C. Barratt, and S. Norman. Linear controller design: limites of performance via convex optimization. *Proceedings of the IEEE*, 78(3):529–574, March 1990.

[6] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, Philadelphia, 1994.

[7] S. Boyd and L. Vandenberghe. *Convex Optimization*. 2002. Available via: `http://www.stanford.edu/~boyd`.

[8] M. Dahleh and I. Diaz-Bobillo. *Control of Uncertain Systems: A Linear Programming Approach*. Prentice Hall, NJ, 1995.

[9] C. E. de Souza and A. Trofino. A linear matrix inequality approach to the design of robust H_2 filters. In L. EL Ghaoui and S.-I. Niculescu, editors, *Advances in Linear Matrix Inequality Methods in Control*, pages 175–185. SIAM, December 1999.

[10] L. El Ghaoui. Personal communication, December 2003.

[11] L. El Ghaoui and G. Calafiore. Robust filtering for discrete-time systems with bounded noise and parametric uncertainty. *IEEE Trans. Auto. Contr.*, 46(7):1084–1089, July 2001.

[12] L. El Ghaoui and J.P. Folcher. Multiobjective robust control of LTI systems subject to unstructured perturbations. *Sys. Contr. Letter*, 28(1):23–30, June 1996.

[13] L. El Ghaoui, R. Nikoukhah, F. Delebecque, and J.-L. Commeau. lmitool, version 2.2. [Online], 2001. Available via: http://robotics.eecs.berkeley.edu/ elghaoui/.

[14] M. Fan, A. Tits, and J. Doyle. Robustness in the presence of joint parametric uncertainty and unmodeled dynamics. *IEEE Trans. Auto. Contr.*, 36:25–38, 1991.

[15] M. Fu, C. de Souza, and Z. Luo. Finite-horizon robust Kalman filter design. *IEEE Trans. Signal Proc.*, 49(9):2103–12, 2001.

[16] M. Fu, C. E. de Souza, and L. Xie. H_∞ estimation for uncertain systems. *Int. J. Rob. Nonlinear Contr.*, 2:87–105, 1992.

[17] J. C. Geromel. Optimal linear filtering under parameter uncertainty. *IEEE Trans. Signal Proc.*, 47:168–175, 1998.

[18] J. C. Geromel, J. Berussou, G. Garcia, and M. C. De Oliveira. H_2 and H_∞ robust filtering for discrete-time linear systems. *SIAM J. Contr. Optim.*, 38(5):1353–1368, 2000.

[19] J. C. Geromel, M. C. De Oliveira, and J. Berussou. Robust filtering of discrete-time linear systems with parameter dependent Lyapunov functions. *SIAM J. Contr. Optim.*, 41(3):700–711, 2002.

[20] A. Ghulchak and A. Rantzer. Robust control under parametric uncertainty via primal-dual convex analysis. *IEEE Trans. Auto. Contr.*, 47(4):632–636, April 2002.

[21] D. Henrion and J. Lasserre. Gloptipoly: Global optimization over polynomials with Matlab and SeDuMi. In *Proc. Conf. Dec. Ctr.*, pages 747–752, 2002.

[22] H. Hindi and S. Boyd. Analysis of linear systems with saturating using convex optimization. In *Proc. Conf. Dec. Ctr.*, pages 903–908, 1998.

[23] T. Hu and Z. Lin. Exact characterization of invariant ellipsoids for linear systems with saturating actuators. *IEEE Trans. Auto. Contr.*, 47:164–169, 2002.

[24] T. Hu, Z. Lin, and B. Chen. An analysis and design method for linear systems subject to actuator saturation and disturbance. *Automatica*, 38:351–359, 2002.

[25] Z. Jarvis-Wloszek, R. Feeley, W. Tan, K. Sun, and A. Packard. Some controls applications of Sum of Squares programming. In *Proc. Conf. Dec. Ctr.*, pages 4676–4681, 2003.

[26] P. P. Khargonekar, I. R. Petersen, and K. Zhou. Robust stabilization of uncertain systems and H_∞ optimal control. *IEEE Trans. Auto. Contr.*, AC-35(3):356–361, 1990.

[27] H. Li and M. Fu. A linear matrix inequality approach to robust H_∞ filtering. *IEEE Trans. Signal Proc.*, 45(9):2338–2350, 1997.

[28] J. Löfberg. Yalmip: Yet another LMI parser, 2003. Available via: `http://www.control.isy.liu.se/~johanl/yalmip.html`.

[29] K. M. Nagpal and P. P. Khargonekar. Filtering and smoothing in an H_∞ setting. *IEEE Trans. Auto. Contr.*, 36(2):152–166, February 1991.

[30] M.P. Newlin and M.P. Young. Mixed mu problems and branch and bound techniques. *Int. J. Rob. Nonlinear Contr.*, 7(2):145–64, 1997.

[31] A. Packard, G. Balas, R. Liu, and J. Shin. Results on worst-case performance assessment. In *Proc. Amer. Conf. Ctr.*, pages 2425–2427, Chicago, Il, 2000.

[32] A. Packard and J. Doyle. Quadratic stability with real and complex perturbations. *IEEE Trans. Auto. Contr.*, AC-35:198–201, 1990.

[33] A. Packard and J. C. Doyle. The complex structured singular value. *Automatica*, 29:71–109, 1993.

[34] A. Packard and P. Pandey. Continuity properties of the real/complex structured singular value. *IEEE Trans. Auto. Contr.*, 38:415–428, 1993.

[35] F. Paganini and A. Giusto. Robust synthesis of dynamic prefilters. In *Proc. Amer. Conf. Ctr.*, pages 1314–1318, June 1997.

[36] R. Palhares and P. Peres. LMI approach to the mixed H_2/H_∞ filtering design for discrete-time uncertain systems. *IEEE Trans. Aeros. Electr. Systems*, 37(1):292–296, 2001.

[37] R.M. Palhares and P.L.D Peres. Optimal filtering schemes for linear discrete-time systems - an LMI approach. *Int. J. Syst. Science*, 29(6):587–593, 1998.

[38] P. Parrilo. On the numerical solution of LMIs derived from the KYP lemma. In *Proc. Conf. Dec. Ctr.*, pages 2334–2338, Phoenix, Ar, 1999.

[39] P. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming*, to appear. available at `http://www.aut.ee.ethz.ch/~parrilo`.

[40] I. R. Petersen and D. C. McFarlane. Optimal guaranteed cost control and filtering for uncertain linear systems. *IEEE Trans. Auto. Contr.*, 39(9):1971–1977, 1994.

[41] I. R. Petersen and D. C. McFarlane. Optimal guaranteed cost filtering for uncertain discrete time systems. *Int. J. Rob. Nonlinear Contr.*, 6(4):267–280, 1996.

[42] I. R. Petersen and A. V. Savkin. *Robust Kalman Filtering for Signals and Systems with Large Uncertainties.* Birkhauser, Boston, 1999.

[43] A. Pinkus. *n-Widths in Approximation Theory (Ergebnisse der Mathematik und ihrer Grenzgebiete).* Berlin: Springer-Verlag, 1985.

[44] E. Polak. *Optimization: Algorithms and Consistent Approximations.* Springer, 1997.

[45] S. Prajna, A. Papachristodoulou, and P. Parrilo. Introducing sostools: A general purpose sum of squares programming solver. In *Proc. Conf. Dec. Ctr.*, pages 741–746, 2002.

[46] A. H. Sayed. A framework for state-space estimation with uncertain models. *IEEE Trans. Auto. Contr.*, 46(7):998–1013, 2001.

[47] C. W. Scherer. Multiobjective H_2/H_∞ control. *IEEE Trans. Auto. Contr.*, 40(6):1054–1062, June 1995.

[48] M.M. Seron, J.A. De Dona, and G.C. Goodwin. Global analytical model predictive control with input constraints. In *Proc. Conf. Dec. Ctr.*, pages 154–159, 2000.

[49] U. Shaked and C. E. de Souza. Robust minimum variance filtering. *IEEE Trans. Signal Proc.*, 43:2474–2483, 1995.

[50] J. F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653, 1999. Version 1.05 available from `http://fewcal.kub.nl/sturm`.

[51] A. Teel. Anti-windup for exponentially unstable linear systems. *Int. J. Rob. Nonlinear Contr.*, 9:701–16, 1999.

[52] Y. Theodor and U. Shaked. Robust discrete-time minimum-variance filtering. *IEEE Trans. Signal Proc.*, 44(2):181–189, 1996.

[53] A. Tits and M. Fan. On the small-μ theorem. *Automatica*, 31(8):1199–1201, 1995.

[54] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, March 1996.

[55] F. Wu and A. Packard. Optimal LQG performance of linear uncertain systems using state-feedback. In *Proc. Amer. Conf. Ctr.*, volume 6, pages 4435–4439, June 1995.

[56] L. Xie and E. de Souza. On robust filtering for linear systems with parameter uncertainty. In *Proc. Conf. Dec. Ctr.*, pages 2087–92, New Orleans, LA, 1995.

[57] L. Xie, E. de Souza, and M. Fu. H_∞ estimation for discrete-time linear uncertain systems. *Int. J. Rob. Nonlinear Contr.*, 1:111–123, 1991.

[58] L. Xie, Y. C. Soh, and C. E. de Souza. Robust kalman filtering for uncertain discrete-time systems. *IEEE Trans. Auto. Contr.*, 39(6):1310–1314, June 1994.

[59] P. Young and J.C. Doyle. Properties of the mixed μ problem and its bounds. *IEEE Trans. Auto. Contr.*, 41(1):155–159, January 1996.

[60] P.M. Young. Controller design with real parametric uncertainty. *Int. Jour. Contr.*, 65(3):469–509, 1996.

[61] P.M. Young, M.P. Young, and J.C. Doyle. Computing bounds for the mixed mu problem. *Int. J. Rob. Nonlinear Contr.*, 5(6):573–590, 1995.

[62] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, Upper Saddle River, New Jersey, 1996.