Continuity properties of the real/complex structured singular value

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Abstract

The structured singular value function (μ) is defined with respect to a given uncertainty set. This function is continuous if the uncertainties are allowed to be complex. However, if some uncertainties are required to be real then it can be discontinuous. It is shown that μ is always upper semicontinuous and conditions are derived under which it is also lower semicontinuous. With these results, the real-parameter robustness problem is re-examined. A related (though not equivalent) problem is formulated, which is always continuous, and relationship between the new problem and the original real μ problem is made explicit. A numerical example and results obtained via this related problem are presented.

1 Introduction

The structured singular value (SSV or μ), introduced in [8], [10], is an important linear algebra tool to study a class of matrix perturbation problems. Specifically, it is useful for obtaining nonconservative bounds for robustness of stability and performance of uncertain linear systems. In developing a model for linear systems it is natural to define a description of uncertainty associated with the model. One possibility is to lump all forms of uncertainty in a single norm-bounded uncertainty. While this may simplify subsequent analysis and design, it may prove to be too conservative especially when performance requirements are stringent. The structured singular value was introduced to take advantage of the fact that in many problems uncertainty can be represented in a structured form, e.g. a block-diagonal form. Algorithms were developed to compute upper and lower bounds for μ and the computed bounds were usually tight enough for practical applications (see [15], [11]). This led to a significant reduction in conservatism over methods which simply lump all uncertainty into a single norm-bounded block.

Initially, the block-diagonal form of uncertainty was assumed to be made up of **complex** blocks. This **complex-μ** approach was reasonable and adequate for many applications and the resulting computational problem was somewhat tractable. However, in many cases it is natural to model uncertainty with real perturbation, e.g., when the real coefficients of a linear differential equation are uncertain. While it is still possible to simply assume that these perturbations are complex and proceed with the analysis, the results can be expected to be conservative. Hence, researchers have focused on **mixed-μ** problems, which allow real perturbations in addition to complex perturbations. This led to new algorithms for computing the SSV when the uncertainty specifications include real perturbations, [9], [20], [21], [12], [24], [7], [2], [23].

It is relatively straightforward to show that the SSV with respect to purely complex perturbations is a continuous function (see Section 4). However, if real perturbations are considered, then the SSV can be discontinuous, and an example of such discontinuity is in [3]. In this example a sequence of stable, 4'th order systems converges, in $\mathcal{H}_\infty$ norm, to a 4'th order, stable system. However, the supremum of the structured singular value of the frequency responses does not converge. This might be a serious problem. In computing the SSV of a given matrix one is almost always computing the SSV of a neighboring matrix, due to finite machine precision. This means that one could end up with misleading results. The focus of this paper is to study these continuity properties in greater detail.

The organization of the paper is as follows. In Section 2 we define the structured singular value function and show that it is upper semicontinuous. Next, in Section 3 we discuss the role of μ in testing the robustness of stability and performance of linear time-invariant systems. We consider **complex** block structures in Section 4 and show that μ is continuous in this case. In Section 5 and 6 we examine mixed block structures (not necessarily complex) and show that μ is continuous if certain assumptions are met. We also derive some conditions under which these assumptions are met. With these results in mind, we re-examine the real-μ problem in Section 7. We formulate a related problem, prove that it is continuous, and make explicit the relationship between the new problem and the original real μ problem. From an engineering point of view, we believe that the new problem is **as relevant** to studying real-perturbation robustness problems as is real-μ, however it is much more practical, given the continuity properties of the new margin. Finally, we present a numerical example and results obtained via this related problem.

2 Definition and Properties

Our notation is standard. $\mathbf{C}^{n\times m}$ and $\mathbf{R}^{n\times m}$ are, respectively, the set of complex and real $n \times m$ matrices. For $M \in \mathbf{C}^{n\times m}$, the maximum singular value of M is denoted by $\bar{\sigma}(M)$, and M^* is the complex conjugate transpose of M. For square matrices $M \in \mathbf{C}^{n\times n}$, $\Lambda(M)$ denotes the spectrum of M, $\rho(M)$ denotes the spectral radius, and $\rho_R(M)$ denotes the *real* spectral radius. The real spectral radius is defined as

$$\rho_R(M) := \max\{|\lambda| : \lambda \in \Lambda(M) \cap \mathbf{R}\}$$

and $\rho_R(M) := 0$ if $\Lambda(M) \cap \mathbf{R} = \phi$ (the empty set). The notation $\|M\|$ will denote any matrix norm, usually the maximum singular value of the matrix.

The structured singular value, [8], is always computed with respect to a specified set $\mathbf{\Delta} \subset \mathbf{C}^{n\times n}$, usually referred to as the "block structure." This is a problem-specific set which depends on the *uncertainty* and *performance objectives* of the problem. For the moment the only assumption about $\mathbf{\Delta}$ is that it is a *closed* subset of $\mathbf{C}^{n\times n}$. Any such closed set will be called an **arbitrary block structure**.

Definition 2.1 *Consider a matrix* $M \in \mathbf{C}^{n\times n}$ *and a closed set* $\mathbf{\Delta} \subset \mathbf{C}^{n\times n}$. *The structured singular value function* $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n\times n} \to \mathbf{R}$ *is defined via*

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\}}, \tag{2.1}$$

unless no $\Delta \in \mathbf{\Delta}$ *makes* $I - M\Delta$ *singular, in which case,* $\mu_{\mathbf{\Delta}}(M) := 0$. ∎

Lemma 2.2 *Given a block structure* $\mathbf{\Delta}$ *which is closed in* $\mathbf{C}^{n\times n}$, *the function* $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n\times n} \to \mathbf{R}$ *is upper semicontinuous, (upper and lower semicontinuity are defined in [18]). Equivalently, for any* $M \in \mathbf{C}^{n\times n}$ *and* $\beta > 0$, *if* $\mu_{\mathbf{\Delta}}(M) < \beta$, *then there is an* $\alpha > 0$ *such that* $\mu_{\mathbf{\Delta}}(X) < \beta$ *for all* X *with* $\|X - M\| < \alpha$.

Proof: Define the set $\mathbf{\Delta}_\beta \subset \mathbf{\Delta}$ by

$$\mathbf{\Delta}_\beta := \left\{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq \frac{1}{\beta}\right\}.$$

By assumption $\mu_{\mathbf{\Delta}}(M) < \beta$. Hence, $\det(I - M\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}_\beta$. Now, for each $\Delta \in \mathbf{\Delta}_\beta$ there exists, by continuity, constants $\alpha_\Delta > 0$ and $\gamma_\Delta > 0$ so that $\det\left(I - X\tilde{\Delta}\right) \neq 0$ for all $X \in \mathbf{C}^{n\times n}$ and $\tilde{\Delta} \in \mathbf{\Delta}$ with $\|X - M\| < \alpha_\Delta$ and $\|\tilde{\Delta} - \Delta\| < \gamma_\Delta$. This forms an open cover of $\mathbf{\Delta}_\beta$. Choose a finite open cover $(\Delta_1, \ldots, \Delta_N), (\alpha_1, \ldots, \alpha_N), (\gamma_1, \ldots, \gamma_N)$. Set $\alpha := \min_{i=1,\ldots,N} \alpha_i$. Note that $\alpha > 0$. Now choose any $\Delta \in \mathbf{\Delta}_\beta$. Then $\|\Delta - \Delta_i\| < \gamma_i$ for some $i \leq N$. Hence, $\det(I - X\Delta) \neq 0$ for all X such that $\|X - M\| < \alpha$ which, by definition, implies that $\mu_{\mathbf{\Delta}}(X) < \beta$. ∎

Remark 2.3 Hence, if $\mu_{\mathbf{\Delta}}(M) = 0$, then $\mu_{\mathbf{\Delta}}$ is continuous at M.

Remark 2.4 The requirement that $\mathbf{\Delta}$ be a closed subset of $\mathbf{C}^{n\times n}$ is necessary. For example, consider $M = 1$ and $\mathbf{\Delta} := [0, 1)$. Clearly, $\mu_{\mathbf{\Delta}}(M) = 0$ since $(1 - \delta) \neq 0$ for any $\delta \in \mathbf{\Delta}$. However, for a perturbed matrix $M_\epsilon := M - \epsilon$ we get $\mu_{\mathbf{\Delta}}(M_\epsilon) = 1 - \epsilon$ for $\epsilon \in (0, 1)$. Hence, $\mu_{\mathbf{\Delta}}$ fails to be upper semicontinuous at M.

In this paper, we often make use of block partitioned matrices. Suppose $M \in \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)}$ is given, and partition it in the obvious way,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{2.2}$$

with $M_{ij} \in \mathbf{C}^{n_i\times n_j}$. Let $\mathbf{\Delta}_1 \subset \mathbf{C}^{n_1\times n_1}$ and $\mathbf{\Delta}_2 \subset \mathbf{C}^{n_2\times n_2}$ be two block structures. Define an augmented block structure $\mathbf{\Delta}$ as

$$\mathbf{\Delta} := \{\text{diag}\,[\Delta_1, \Delta_2] : \Delta_i \in \mathbf{\Delta}_i\} \tag{2.3}$$

which is compatible, in dimension, with block-partitioned matrices defined above. Also define the sets:

$$\mathbf{B}_i := \{\Delta_i \in \mathbf{\Delta}_i : \bar{\sigma}(\Delta_i) \leq 1\}$$

and

$$\mathbf{B}_{\mathbf{\Delta}} := \{\text{diag}\,[\Delta_1, \Delta_2] : \Delta_i \in \mathbf{B}_i\}.$$

Note that we can compute μ with respect to three block structures. The notation we will use to keep track of this is as follows: $\mu_1(\cdot)$ is with respect to $\mathbf{\Delta}_1$, $\mu_2(\cdot)$ is with respect to $\mathbf{\Delta}_2$, and $\mu_{\mathbf{\Delta}}(\cdot)$ is with respect to $\mathbf{\Delta}$. In view of this, $\mu_1(M_{11})$, $\mu_2(M_{22})$ and $\mu_{\mathbf{\Delta}}(M)$ all make sense, though for instance, $\mu_1(M)$ does not. Note that the inequality

$$\mu_{\mathbf{\Delta}}(M) \geq \max\{\mu_1(M_{11}), \mu_2(M_{22})\}, \tag{2.4}$$

is immediate from the definition of μ. This fact, and upper semicontinuity of μ imply the next lemma.

Lemma 2.5 *Suppose $M(\epsilon) \in \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)}$ is a matrix-valued function of the scalar parameter ϵ defined by*

$$M(\epsilon) := \begin{bmatrix} M_{11} & M_{12}(\epsilon) \\ M_{21}(\epsilon) & M_{22}(\epsilon) \end{bmatrix}, \tag{2.5}$$

where $M_{ij} \in \mathbf{C}^{n_i\times n_j}$, and

$$\lim_{\epsilon\to 0} M(\epsilon) = \begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

Also, suppose $\mathbf{\Delta}$ is a block structure as defined in (2.3). Then

$$\lim_{\epsilon\to 0} \mu_{\mathbf{\Delta}}(M(\epsilon)) = \mu_1(M_{11}).$$

Proof: By (2.4), for any ϵ we must have

$$\mu_{\mathbf{\Delta}}(M(\epsilon)) \geq \max\{\mu_1(M_{11}), \mu_2(M_{22}(\epsilon))\}. \tag{2.6}$$

Since $\mu_{\mathbf{\Delta}}$ is upper semicontinuous, given any $\delta > 0$ there is an $\epsilon_* > 0$ such that for all ϵ with $|\epsilon| < \epsilon_*$, we have

$$\mu_{\mathbf{\Delta}}(M(\epsilon)) < \mu_{\mathbf{\Delta}}(M(0)) + \delta = \mu_1(M_{11}) + \delta.$$

This, together with (2.6), implies the desired result. ∎

In many control problems, it is possible to represent uncertainty as *linear fractional.* Let $\Delta_2 \in \mathbf{\Delta}_2$. If $I - M_{22}\Delta_2$ is invertible, then the **linear fractional transformation** (LFT), denoted by $F_l(M, \Delta_2)$, is **well-posed**, and defined as

$$F_l(M, \Delta_2) := M_{11} + M_{12}\Delta_2(I - M_{22}\Delta_2)^{-1}M_{21}. \tag{2.7}$$

This is shown in feedback form in Figure 1.

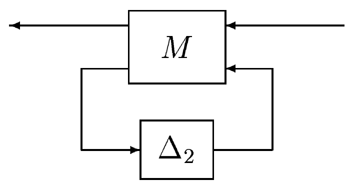

Figure 1: Linear Fractional Transformation

The subscript l on F_l pertains to fact that the "lower" loop of M is closed by Δ_2. An analogous formula describes $F_u(M, \Delta_1)$, which is the resulting matrix obtained by closing the "upper" loop of M with a matrix $\Delta_1 \in \mathbf{\Delta}_1$.

In this formulation the matrix M_{11} is considered to be "nominal" and $\Delta_2 \in \mathbf{B}_2$ is viewed as a norm bounded perturbation from an allowable perturbation set, $\mathbf{\Delta}_2$. The matrices M_{12}, M_{21}, and M_{22} together with the formula F_l reflect prior knowledge of how the unknown perturbation affects the nominal matrix, M_{11}. This type of uncertainty encompasses many other cases considered by researchers, including real parameter variations in differential equation models.

As the "perturbation" Δ_2 deviates from zero, the matrix $F_l(M, \Delta_2)$ deviates from M_{11}. The following theorem shows that it is possible to nonconservatively bound the quantity $\mu_1(F_l(M, \Delta_2))$ using $\mu_{\mathbf{\Delta}}(M)$.

Theorem 2.6 *For a block-partitioned matrix $M \in \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)}$ defined in (2.2), and the uncertainty set $\mathbf{\Delta}$ in defined (2.3), the following equivalence holds:*

$$\mu_{\mathbf{\Delta}}(M) < 1 \quad \Leftrightarrow \quad \begin{array}{c} \mu_2(M_{22}) < 1 \\ \max_{\Delta_2 \in \mathbf{B}_2} \mu_1(F_l(M, \Delta_2)) < 1 \end{array}$$

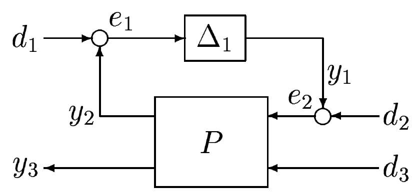

Figure 2: Linear Fractional Uncertainty Model

Proof: The proof follows from the definition of $\mu_{\mathbf{\Delta}}(M)$ and Schur formulas for the determinant of block partitioned matrices, [9] and [19]. Note that

$$\sup_{\Delta_2 \in \mathbf{B}_2} \mu_1 (F_l (M, \Delta_2)) = \max_{\Delta_2 \in \mathbf{B}_2} \mu_1 (F_l (M, \Delta_2))$$

because $\mathbf{B}_2$ is compact and μ_1 is upper semicontinuous. ∎

This theorem forms the basis for all applications of the structured singular value in linear systems theory, whether from a frequency domain, state space, or Lyapunov point of view. Note that the result holds for general block structures $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$.

3 Robustness tests with μ

The structured singular value can be used to compute robustness margins for a linear system with linear fractional uncertainty. The simplest application involves constant, but unknown, structured perturbations. Specifically, suppose that $P(s)$ is a rational, proper matrix, of size $(n_1 + n_2) \times (n_1 + n_2)$ and block structures $\mathbf{\Delta}_1 \subset \mathbf{C}^{n_1 \times n_1}$ and $\mathbf{\Delta}_2 \subset \mathbf{C}^{n_2 \times n_2}$ are given. Partition $P(s)$ as

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}.$$

Since P is rational and proper, the following limit is well-defined

$$P(\infty) = \begin{bmatrix} P_{11}(\infty) & P_{12}(\infty) \\ P_{21}(\infty) & P_{22}(\infty) \end{bmatrix} := \lim_{s \to \infty} P(s).$$

For $\Delta_1 \in \mathbf{\Delta}_1$, consider the interconnection shown in Figure 2.

For any $\Delta_1 \in \mathbf{B}_1$, the closed-loop system is said to be:

- **well-posed** if $\det (I - P_{11}(\infty)\Delta_1) \neq 0$. This is the necessary and sufficient condition that all closed-loop transfer functions in Figure 2 be proper.

- **stable** if all closed-loop transfer functions in Figure 2 are analytic in the closed right-half-plane.

Theorem 3.1 *Suppose that $P(s)$ has all of its poles in the open left-half plane. Let $\beta > 0$. Then*

1. *For all $\Delta_1 \in \mathbf{\Delta}_1$ with $\bar{\sigma}(\Delta_1) \leq \beta$, the perturbed closed-loop system is well-posed and stable if and only if*

$$\max \left\{ \sup_{\mathrm{Re}(s) \geq 0} \mu_1 \left(P_{11}(s) \right), \mu_1 \left(P_{11}(\infty) \right) \right\} < \frac{1}{\beta}.$$

2. *For all $\Delta_1 \in \mathbf{\Delta}_1$ with $\bar{\sigma}(\Delta_1) \leq \beta$, the perturbed closed-loop system is well-posed, stable and*

$$\max \left\{ \sup_{\mathrm{Re}(s) \geq 0} \mu_2 \left[F_u \left(P(s), \Delta_1 \right) \right], \mu_2 \left[F_u \left(P(\infty), \Delta_1 \right) \right] \right\} < \frac{1}{\beta}$$

if and only if

$$\max \left\{ \sup_{\mathrm{Re}(s) \geq 0} \mu_{\mathbf{\Delta}} \left(P(s) \right), \mu_{\mathbf{\Delta}} \left(P(\infty) \right) \right\} < \frac{1}{\beta}.$$

Proof: The proof follows from the definitions of μ, well-posedness, and stability. ∎

Remark 3.2 Although the structured singular value is **not necessarily a norm**, we introduce the following notation: for a proper, rational matrix P, analytic in the closed-right-half-plane, and a block structure $\mathbf{\Delta}$ of appropriate dimensions, define

$$\|P\|_{\mathbf{\Delta}} := \max \left\{ \sup_{\mathrm{Re}(s) \geq 0} \mu_{\mathbf{\Delta}} \left(P(s) \right), \mu_{\mathbf{\Delta}} \left(P(\infty) \right) \right\}.$$

Remark 3.3 In Section 6, the right-half plane supremums will be replaced (equivalently) with imaginary-axis supremums.

Other robustness tests may be formulated with μ, including linear, time-invariant Δ_1 and gap/graph topology uncertainty, [13], [16], and different induced norms on the perturbations, [1].

4 Complex Block Structures

Strong continuity properties of the SSV can be deduced for certain special block structures which are often encountered in control problems. For example, block diagonal uncertainties were introduced in [22]. These are useful when several independent components, that together make up a larger system, are uncertain, with the uncertainty taking the form of unmodelled dynamics. Also in this setting, it is possible to put $\mathcal{H}_\infty$ norm constraints on perturbed closed-loop disturbance/error transfer functions. This framework gives rise to a special type of block structure, called a **complex block structure**.

Defining a complex block structure involves specifying three things; the type of each block, the total number of blocks, and their dimensions. We consider two types of blocks: *repeated scalar* and *full* blocks. Two nonnegative integers, S and F, represent the number of *repeated scalar* blocks and the number of *full* blocks, respectively. To keep track of their dimensions, we introduce positive integers $r_1, \ldots, r_S;\ m_1, \ldots, m_F$. The i'th repeated scalar block is $r_i \times r_i$, while the j'th full block is $m_j \times m_j$. For consistency among all the dimensions, we must have $\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$. With those integers given, define $\mathbf{\Delta} \subset \mathbf{C}^{n\times n}$ as

$$\mathbf{\Delta} \ = \left\{\operatorname{diag}\left[\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}, \Delta_1, \ldots, \Delta_F\right] : \delta_i \in \mathbf{C}, \Delta_j \in \mathbf{C}^{m_j\times m_j}\right\} \tag{4.1}$$

Associated with $\mathbf{\Delta}$ is the unit ball $\mathbf{B_\Delta} := \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq 1\}$ and the set of unitary elements $\mathbf{Q_\Delta} := \{Q \in \mathbf{\Delta} : Q^* Q = I_n\}$.

For these complex block structures, an alternative expression for $\mu_{\mathbf{\Delta}}(M)$ follows from Definition 2.1.

Lemma 4.1 *For $M \in \mathbf{C}^{n\times n}$ and a complex block structure $\mathbf{\Delta}$ defined in (4.1),*

$$\mu_{\mathbf{\Delta}}(M) = \max_{\Delta \in \mathbf{B_\Delta}} \rho(M\Delta).$$

Since $\mathbf{B_\Delta}$ is compact, and $\rho(\cdot)$ is continuous, we have:

Corollary 4.2 *If $\mathbf{\Delta}$ is a complex block structure as defined in (4.1), then $\mu_{\mathbf{\Delta}}$ is continuous.*

Additional results in this paper rely heavily on the following two well-known results from complex analysis, which we state without proof. The first is Rouche's theorem, [18].

Theorem 4.3 *Let Γ be a simple closed contour in the complex plane, $\mathbf{C}$. Let f and g be functions which are analytic inside and on Γ. If $|g(z)| < |f(z)|$ on Γ, then f and $f+g$ have the same number of roots (zeros) inside Γ.*

This is used in proving the next lemma, which is the well known result stating that the roots of a polynomial are continuous functions of the coefficients of the polynomial (even the leading coefficients that are zero).

Lemma 4.4 *Let $f(z) = \sum_{i=0}^{n} a_i z^i$ be an n'th order polynomial, $a_n \neq 0$. Let $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n$ be the roots of f. For any $\epsilon > 0$ and any integer $m > 0$, there exists a $\delta_{m,\epsilon} > 0$ such that if $g(z)$, defined by*

$$g(z) = \sum_{i=0}^{m} b_i z^i$$

has coefficients $b_i \in \mathbf{C}$ *which satisfy* $|b_i| < \delta_{m,\epsilon}$*, then there are* n *roots of* $f+g$*, labeled* $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n$ *that satisfy* $|\bar{z}_i - \tilde{z}_i| < \epsilon$*.*

An important result from [8] concerning complex block structures is that the maximum in Lemma 4.1 is achieved on the unitary elements of $\mathbf{\Delta}$. The proof of this uses Theorem 4.3 and Lemma 4.4.

Lemma 4.5 *For* $M \in \mathbf{C}^{n\times n}$ *and a complex block structure* $\mathbf{\Delta}$ *defined in (4.1),*

$$\mu_{\mathbf{\Delta}}(M) = \max_{Q \in \mathbf{Q}_{\mathbf{\Delta}}} \rho(MQ).$$

This fact is important since it reduces some of the computational burden in computing lower bounds for μ. Also, this has implications on subharmonicity and maximum modulus results which are studied in [4]. In [14], Lemma 4.5 was extended to linear fractional transformations in case both $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$, and consequently $\mathbf{\Delta}$, are **complex block structures**.

Lemma 4.6 *Let* $M \in \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)}$ *be a block-partitioned matrix, as defined in (2.2). Also, let* $\mathbf{\Delta}_1 \subset \mathbf{C}^{(n_1)\times(n_1)}$ *and* $\mathbf{\Delta}_2 \subset \mathbf{C}^{(n_2)\times(n_2)}$ *be complex block structures as in (4.1). If* $\mu_2(M_{22}) < 1$*, then*

$$\max_{Q_2 \in \mathbf{Q}_2} \mu_1\left(F_l(M, Q_2)\right) = \max_{\Delta_2 \in \mathbf{B}_2} \mu_1(F_l\left(M, \Delta_2\right)).$$

We shall see in Section 6 that this lemma simplifies certain robustness tests.

5 General Continuity Lemmas

Consider matrices partitioned as in (2.2), and augmented block structures defined in (2.3). Assume that $\mathbf{\Delta}_2$ is a **complex** block structure, as defined in (4.1).

Lemma 5.1 *Consider* M *and* $\mathbf{\Delta}$ *defined in (2.2)–(2.3) where* $\mathbf{\Delta}_2 \subset \mathbf{C}^{n_2\times n_2}$ *is a complex block structure and* $\mathbf{\Delta}_1$ *is any closed subset of* $\mathbf{C}^{n_1\times n_1}$*. If* $\mu_1(M_{11}) < \mu_{\mathbf{\Delta}}(M) =: \beta$ *then* $\mu_{\mathbf{\Delta}}$ *is lower semicontinuous at* M*. Equivalently, for any* $\epsilon > 0$ *there is an* $\alpha > 0$ *such that* $\mu_{\mathbf{\Delta}}(X) > \beta - \epsilon$ *for all* X *such that* $\|X - M\| < \alpha$*.*

Proof: Since $\mu_{\mathbf{\Delta}}(M) = \beta$, there is a $\bar{\Delta}_{1,2} = \operatorname{diag}\left[\bar{\Delta}_1, \bar{\Delta}_2\right] \in \mathbf{\Delta}$ with $\bar{\sigma}\left(\bar{\Delta}_{1,2}\right) = \frac{1}{\beta}$, and $\det\left(I - M\bar{\Delta}_{1,2}\right) = 0$. For a given $\delta > 0$ define the set $\mathbf{X}_\delta \subset \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)}$ by

$$\mathbf{X}_\delta := \{X : \|X - M\| \leq \delta\}.$$

Assume that each $X \in \mathbf{X}_\delta$ is partitioned like M, i.e., $X_{ij} \in \mathbf{C}^{n_i \times n_j}$. Now, $\det\left(I - M_{11}\bar{\Delta}_1\right) \neq 0$ since $\mu_1(M_{11}) < \beta$ and $\bar{\sigma}\left(\bar{\Delta}_1\right) \leq \frac{1}{\beta}$. Hence, by continuity, there is a $\delta > 0$ such that $\det\left(I - X_{11}\bar{\Delta}_1\right) \neq 0$ for all $X \in \mathbf{X}_\delta$. For all such X and all $\Delta_2 \in \mathbf{\Delta}_2$,

$$\det\left(I - X\begin{bmatrix} \bar{\Delta}_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}\right) = \det\left(I - X_{11}\bar{\Delta}_1\right)\det\left(I - F_u\left(X, \bar{\Delta}_1\right)\Delta_2\right).$$

The above expression is equal to 0 at $X = M$ and $\Delta_2 = \bar{\Delta}_2$, so with $\bar{\sigma}\left(\bar{\Delta}_2\right) \leq \frac{1}{\beta}$, it is clear that $\mu_2\left(F_u\left(M, \bar{\Delta}_1\right)\right) \geq \beta$. Now, define the function g on the closed set $\mathbf{X}_\delta$ as follows:

$$g(X) := \mu_2\left(F_u\left(X, \bar{\Delta}_1\right)\right).$$

For all $X \in \mathbf{X}_\delta$, the argument $F_u\left(X, \bar{\Delta}_1\right)$ is well defined, and a continuous function of X. Also, since $\mathbf{\Delta}_2$ is a **complex block structure**, the function μ_2 is a continuous function. Hence, the function g is continuous on the closed ball around M. Therefore, since $g(M) \geq \beta$, there is an $\alpha \in (0, \delta]$ such that for all X , $\|X - M\| < \alpha$, $g(X) > \beta - \epsilon$. So, for every X, with $\|X - M\| < \alpha$ there is a $\Delta_{2X} \in \mathbf{\Delta}_2$ with $\bar{\sigma}(\Delta_{2X}) < \frac{1}{\beta - \epsilon}$ and $\det\left(I - F_u\left(X, \bar{\Delta}_1\right)\Delta_{2X}\right) = 0$. This implies that $\det\left(I - X \cdot \mathrm{diag}\left[\bar{\Delta}_1, \Delta_{2X}\right]\right) = 0$. Since $\bar{\sigma}\left(\bar{\Delta}_1\right) \leq \frac{1}{\beta} < \frac{1}{\beta - \epsilon}$, it is clear that $\mu_{\mathbf{\Delta}}(X) > \beta - \epsilon$, as desired. ∎

Using the above notation, several corollaries are immediate consequences of Lemma 2.2 and Lemma 5.1.

Corollary 5.2 *If $\mu_{\mathbf{\Delta}}(M) > \mu_1(M_{11})$, then $\mu_{\mathbf{\Delta}}$ is continuous at M.*

Proof: The assumption implies that $\mu_{\mathbf{\Delta}}$ is lower semicontinuous at M. Hence, $\mu_{\mathbf{\Delta}}$ is continuous at M since it is always upper semicontinuous. ∎

Corollary 5.3 *If $\mu_{\mathbf{\Delta}}(M) > \mu_1(M_{11})$, there exists an $\epsilon > 0$ such that if $X \in \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)}$, and $\|X - M\| < \epsilon$, then $\mu_{\mathbf{\Delta}}(X) > \mu_1(X_{11})$.*

Proof: By Corollary 5.2, $\mu_{\mathbf{\Delta}}$ is continuous at M, and by Lemma 2.2, μ_1 is upper semicontinuous at M_{11}. ∎

Corollary 5.4 *If $\mu_{\mathbf{\Delta}}(M) > \mu_1(M_{11})$, there exists an $\epsilon > 0$ such that $\mu_{\mathbf{\Delta}}$ is continuous on the set*

$$\left\{X \in \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)} : \|X - M\| < \epsilon\right\}.$$

Corollary 5.5 *If $\mu_1(M_{11}) = 0$, then $\mu_{\mathbf{\Delta}}$ is continuous at M.*

Proof: Clearly $\mu_{\mathbf{\Delta}}(M) \geq 0$. If $\mu_{\mathbf{\Delta}}(M) > 0$ then also $\mu_{\mathbf{\Delta}}(M) > \mu_1(M_{11})$ and hence continuous. If $\mu_{\mathbf{\Delta}}(M) = 0$ then upper semicontinuity and the fact the $\mu_{\mathbf{\Delta}}(X) \geq 0$ for any X provides continuity. ∎

Corollary 5.6 *If $\mu_{\mathbf{\Delta}}(M) = \mu_2(M_{22})$, then $\mu_{\mathbf{\Delta}}$ is continuous at M.*

Proof: Suppose $\mu_{\mathbf{\Delta}}(M) = \mu_2(M_{22}) = \beta$ for some $\beta > 0$. Then, there is a $\bar{\Delta}_2 \in \mathbf{\Delta}_2$ such that $\bar{\sigma}\left(\bar{\Delta}_2\right) = \frac{1}{\beta}$ and $\det\left(I - M_{22}\bar{\Delta}_2\right) = 0$. Also, μ_2 is continuous since $\mathbf{\Delta}_2$ is a complex block structure. Hence, given any $\epsilon > 0$ there is an $\alpha > 0$ such that $\mu_2(\bar{M}_{22}) > \mu_2(M_{22}) - \epsilon$ for all $\|\bar{M}_{22} - M_{22}\| < \alpha$. Therefore, for all $\|X - M\| < \alpha$, we have $\mu_{\mathbf{\Delta}}(X) \geq \mu_2(X_{22}) > \mu_2(M_{22}) - \epsilon = \mu_{\mathbf{\Delta}}(M) - \epsilon$. Hence, $\mu_{\mathbf{\Delta}}$ is continuous at M. ∎

Now we consider situations that exploit Lemma 5.1 via the condition $\mu_{\mathbf{\Delta}}(M) > \mu_1(M_{11})$. In the first case this condition is guaranteed by imposing certain rank conditions on the blocks of the matrix M.

Lemma 5.7 *Consider M and $\mathbf{\Delta}$ defined in (2.2)–(2.3). Suppose that $\mathbf{\Delta}_2 = \mathbf{C}^{n_2 \times n_2}$ (i.e., a single full complex block), and that M_{21} and M_{12} are, respectively, full column rank and full row rank (this implicitly requires $n_2 \geq n_1$). Then $\mu_{\mathbf{\Delta}}(M) > \mu_1(M_{11})$ and, consequently, $\mu_{\mathbf{\Delta}}$ is continuous on some open neighborhood containing M.*

Proof: If $\mu_1(M_{11}) = 0$ then $\mu_{\mathbf{\Delta}}$ is continuous by Corollary 4.3. Hence, assume $\mu_1(M_{11}) > 0$. Let $\bar{\Delta}_1 \in \mathbf{\Delta}_1$, with $\det\left(I - M_{11}\bar{\Delta}_1\right) = 0$ and $\bar{\sigma}\left(\bar{\Delta}_1\right) = \frac{1}{\mu_1(M_{11})}$. Then there exists a $x \in \mathbf{C}^{n_1}$, $x \neq 0$ such that $\left(I - M_{11}\bar{\Delta}_1\right)x = 0$. Note that $\bar{\Delta}_1 x \neq 0$. Let M_{12}^R and M_{21}^L be some right and left inverses of M_{12} and M_{21}, respectively. For small $\epsilon \geq 0$, define $\Delta_1(\epsilon) := (1-\epsilon)\bar{\Delta}_1$, and

$$K_\epsilon := \frac{\epsilon}{(1-\epsilon)x^*\bar{\Delta}_1^*\bar{\Delta}_1 x} M_{12}^R M_{11}\bar{\Delta}_1 x x^* \bar{\Delta}_1^* M_{21}^L.$$

Define $\Delta_2(\epsilon) := \left(I + K_\epsilon M_{22}\right)^{-1} K_\epsilon$. It is clear that there is a $\bar{\epsilon} > 0$ such that for any ϵ with $0 \leq \epsilon \leq \bar{\epsilon}$, the functions $\Delta_i(\epsilon)$ are well defined, continuous functions, and $I - M_{22}\Delta_2(\epsilon)$ is invertible. Also, for all such ϵ, the matrix $I - F_l(M, \Delta_2(\epsilon))\,\Delta_1(\epsilon)$ is singular. Expand this via Schur formulas, so that for all such ϵ, the matrix

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \Delta_1(\epsilon) & 0 \\ 0 & \Delta_2(\epsilon) \end{bmatrix}$$

is singular. Now, by choosing $\epsilon > 0$, but small enough so that $\bar{\sigma}(\Delta_2(\epsilon)) < \frac{1}{\mu_1(M_{11})}$, it is clear that $\mu_{\mathbf{\Delta}}(M) > \mu_1(M_{11})$. ∎

Another way to use the result in Lemma 5.1 is as follows. Let $P \in \mathbf{C}^{n\times n}$ be given, along with invertible matrices $W_L, W_R \in \mathbf{\Delta}_2$. Define $M \in \mathbf{C}^{2n\times 2n}$ as

$$M := \begin{bmatrix} P & PW_R \\ W_L P & W_L P W_R \end{bmatrix}. \tag{5.1}$$

Lemma 5.8 *Consider $M \in \mathbf{C}^{2n\times 2n}$ defined in (5.1). Suppose $\mathbf{\Delta}_2 \subset \mathbf{C}^{n\times n}$ is a complex block structure and that $\mathbf{\Delta}_1 := \mathbf{R}^{n\times n} \bigcap \mathbf{\Delta}_2$. If $\mu_2(P) > 0$, then $\mu_1(P) < \mu_{\mathbf{\Delta}}(M)$ and $\mu_{\mathbf{\Delta}}$ is continuous on some open neighborhood containing M.*

Proof: Assume, without loss in generality, that $\mu_1(P) > 0$. Consider a $\Delta := \mathrm{diag}[\Delta_1, \Delta_2]$, where $\Delta_1 \in \mathbf{\Delta}_1$ and $\Delta_2 \in \mathbf{\Delta}_2$. Then

$$\det(I - M\Delta) = \det(I - P(\Delta_1 + W_L \Delta_2 W_R)).$$

Let $\bar{\Delta}_1 \in \mathbf{\Delta}_1$, with $\bar{\sigma}\left(\bar{\Delta}_1\right) = \frac{1}{\mu_1(P)}$, and $\det\left(I - P\bar{\Delta}_1\right) = 0$. Define the function

$$f(\Delta_1, \delta) := \det\left(I - P\left(\Delta_1 + W_L\left[\delta W_L^{-1}\bar{\Delta}_1 W_R^{-1}\right] W_R\right)\right).$$

For fixed Δ_1, we can consider $f(\Delta_1, \delta)$ to be a polynomial in the complex variable δ. Also, the coefficients of this polynomial depend continuously on Δ_1. Note that $f\left(\bar{\Delta}_1, 0\right) = 0$, and $f\left(\bar{\Delta}_1, -1\right) = 1$. Hence, the polynomial $f\left(\bar{\Delta}_1, \delta\right)$ is a nontrivial polynomial in the complex variable δ. For any $\epsilon > 0$, there is a $\gamma > 0$ such that, for each $\Delta_1 \in \mathbf{\Delta}_1$, satisfying $\|\Delta_1 - \bar{\Delta}_1\| < \gamma$, there is a $\delta \in \mathbf{C}$, with $|\delta| < \epsilon$ with $f(\Delta_1, \delta) = 0$. In particular, choose

$$\epsilon = \frac{1}{2\,\bar{\sigma}\left(W_L^{-1}\right)\bar{\sigma}\left(W_R^{-1}\right)}$$

Now, choose α such that $1 - \gamma/\bar{\sigma}\left(\bar{\Delta}_1\right) < \alpha < 1$, and $\delta \in \mathbf{C}, |\delta| < \epsilon$ such that with $\Delta_1 := \alpha\bar{\Delta}_1$ and $\Delta_2 := \delta W_L^{-1}\bar{\Delta}_1 W_R^{-1} \in \mathbf{\Delta}_2$, we obtain $\det(I - M\mathrm{diag}[\Delta_1, \Delta_2]) = 0$. Since both $\bar{\sigma}(\Delta_1), \bar{\sigma}(\Delta_2) < \frac{1}{\mu_1(P)}$, this implies that $\mu_{\mathbf{\Delta}}(M) > \mu_1(P)$ as desired. ∎

As a special case of the previous lemma, consider $W_L = W_R = \sqrt{\epsilon} I$.

Lemma 5.9 *Define*

$$M(\epsilon, P) := \begin{bmatrix} P & \sqrt{\epsilon}P \\ \sqrt{\epsilon}P & \epsilon P \end{bmatrix},$$

and the function f as

$$f(\epsilon, P) := \mu_{\mathbf{\Delta}}(M(\epsilon, P)).$$

Then, the function $f : (0, \infty) \times \mathbf{C}^{n\times n} \to \mathbf{R}$ is continuous, $f(\epsilon, P) \geq \mu_1(P)$, and

$$\lim_{\epsilon \to 0} f(\epsilon, P) = \mu_1(P).$$

Proof: Immediate from Lemma 5.8 and 2.5. ∎

A similar result is obtained by augmenting the real uncertainty with imaginary uncertainty.

Lemma 5.10 *Assume that* $\mathbf{\Delta}_1 = \mathbf{R}^{n\times n} \bigcap \mathbf{\Delta}_C$ *for some complex block structure* $\mathbf{\Delta}_C$*, and let* $\mathbf{\Delta}_2 := \mathbf{\Delta}_1$*. Define*

$$M(\epsilon, P) := \begin{bmatrix} P & \sqrt{\epsilon}P \\ j\sqrt{\epsilon}P & j\epsilon P \end{bmatrix},$$

$\mathbf{\Delta} := \{\mathrm{diag}\,[\Delta_1, \Delta_2] : \Delta_i \in \mathbf{\Delta}_i\}$, *and*

$$f(\epsilon, P) := \mu_{\mathbf{\Delta}}\,(M(\epsilon, P)).$$

Then, the function $f : (0, \infty) \times \mathbf{C}^{n\times n} \to \mathbf{R}$ *is continuous,* $f(\epsilon, P) \geq \mu_1(P)$*, and*

$$\lim_{\epsilon\to 0} f(\epsilon, P) = \mu_1(P).$$

Sketch of Proof: For any $\epsilon > 0$ and matrix P,

$$\det\,(I - M(\epsilon, P)\mathrm{diag}\,[\Delta_1, \Delta_2]) = \det\,(I - P\,[\Delta_1 + j\epsilon\Delta_2])\,.$$

This can be thought of as a polynomial in the "complex variable" $\Delta_1 + j\epsilon\Delta_2$. By Lemma 4.4, these roots vary continuously with P. Since $\epsilon > 0$, the roots (as real variables Δ_1 and Δ_2) of the polynomial

$$\det\,(I - P\,[\Delta_1 + j\epsilon\Delta_2]) = 0$$

vary continuously with P and ϵ. Hence if $\mu_{\mathbf{\Delta}}\,(M\,(\epsilon, P)) > \beta$, there is a $\gamma > 0$ such that if $\left\|P - \tilde{P}\right\| < \gamma$, and $|\epsilon - \tilde{\epsilon}| < \gamma$, then $\mu_{\mathbf{\Delta}}\left(M\left(\tilde{\epsilon}, \tilde{P}\right)\right) > \beta$. ∎

6 Real/Complex μ Problems

In the previous sections, we have defined **arbitrary** and **complex** block structures. A closed subset $\mathbf{\Delta} \subset \mathbf{R}^{n\times n}$ is called a **real block structure** if for all $\alpha \in \mathbf{R}$, $\alpha \neq 0$,

$$\mathbf{\Delta} = \{\alpha\Delta : \Delta \in \mathbf{\Delta}\}\,. \tag{6.2}$$

Suppose $\mathbf{\Delta}_1 \subset \mathbf{R}^{n_1\times n_1}$ is a real block structure, and define $\mathbf{B}_1$ and $\mathbf{Q}_1$ as follows:

$$\mathbf{B}_1 = \mathbf{Q}_1 := \{\Delta_1 \in \mathbf{\Delta}_1 : \bar{\sigma}\,(\Delta_1) \leq 1\}\,.$$

Let $\mathbf{\Delta}_2 \subset \mathbf{C}^{n_2\times n_2}$ be a complex block structure, with associated sets $\mathbf{B}_2$ and $\mathbf{Q}_2$. Now $\mathbf{\Delta}$, $\mathbf{B}_{\mathbf{\Delta}}$, and $\mathbf{Q}_{\mathbf{\Delta}}$ are the obvious diagonal augmentations involving $\mathbf{\Delta}_1$, $\mathbf{\Delta}_2$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$. Lemma 4.5, which provides an alternative characterization of $\mu_{\mathbf{\Delta}}\,(M)$ for complex block structures, was extended in [24] for such mixed block structures. As stated below, the next theorem is a slight extension of Theorem 1 in [24]. However, the proof is nearly identical and is omitted.

Lemma 6.1 *Consider an augmented block structure* $\mathbf{\Delta}$ *where* $\mathbf{\Delta}_1$ *is a real block structure defined in (6.2) and* $\mathbf{\Delta}_2$ *is a complex block structure. Then for* $M \in \mathbf{C}^{n \times n}$

$$\mu_{\mathbf{\Delta}}(M) = \max_{Q \in \mathbf{Q_\Delta}} \rho_R(MQ).$$

Similarly, Lemma 4.6 can be extended to include real block structures.

Lemma 6.2 *Let* $M \in \mathbf{C}^{(n_1+n_2)\times(n_1+n_2)}$ *be a block-partitioned matrix, as defined in (2.2), Let* $\mathbf{\Delta}_1 \subset \mathbf{C}^{n_1 \times n_1}$ *be a real block structure as defined in (6.2) and* $\mathbf{\Delta}_2 \subset \mathbf{C}^{n_2 \times n_2}$ *be complex block structure. Suppose* $\mu_2(M_{22}) < 1$*. Then*

$$\max_{\Delta_2 \in \mathbf{B}_2} \mu_1(F_l(M, \Delta_2)) = \max_{Q_2 \in \mathbf{Q}_2} \mu_1(F_l(M, Q_2)).$$

Proof: Suppose $\max_{\Delta_2 \in \mathbf{B}_2} \mu_1(F_l(M, \Delta_2)) = 1$. Then Theorem 2.6 implies that $\mu_{\mathbf{\Delta}}(M) \geq 1$. We first show that $\mu_{\mathbf{\Delta}}(M) = 1$. Suppose $\mu_{\mathbf{\Delta}}(M) > 1$. Then,

$$\mu_1(M_{11}) \leq \max_{\Delta_2 \in \mathbf{B}_2} \mu_1(F_l(M, \Delta_2)) = 1 < \mu_{\mathbf{\Delta}}(M)$$

implying that $\mu_{\mathbf{\Delta}}(M) > \mu_1(M_{11})$ and that $\mu_{\mathbf{\Delta}}(M)$ is continuous at M (by Lemma 5.1). Since $\mu_{\mathbf{\Delta}}(M) > 1$, for some nonzero $\alpha < 1$ the matrix

$$M_\alpha := \begin{bmatrix} \alpha M_{11} & \alpha M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

satisfies $\mu_{\mathbf{\Delta}}(M_\alpha) > 1$. This follows from the continuity of $\mu_{\mathbf{\Delta}}$ at M. Then, Theorem 2.6 implies that

$$\max_{\Delta_2 \in \mathbf{B}_2} \mu_1(F_l(M_\alpha, \Delta_2)) \geq 1.$$

But using the expression for M_α we get

$$\max_{\Delta_2 \in \mathbf{B}_2} \mu_1(F_l(M, \Delta_2)) \geq \frac{1}{\alpha} > 1,$$

which is a contradiction. Hence, $\mu_{\mathbf{\Delta}}(M) = 1$ and, by definition, there is a $\Delta = \text{diag}(\Delta_1, \Delta_2) \in \mathbf{\Delta}$ such that $\det(I - M\Delta) = 0$. But then, by Lemma 6.1, there is a $Q = \text{diag}(Q_1, Q_2) \in \mathbf{Q_\Delta}$ such that $\det(I - MQ) = 0$. Since $\mu_2(M_{22}) < 1$ the matrix $(I - M_{22}Q_2)$ is invertible and

$$\det(I - MQ) = \det(I - M_{22}Q_2)\det(I - F_l(M, Q_2)Q_1) = 0$$

implying that $\mu_1(F_l(M, Q_2)) = 1$. ∎

Remark 6.3 In this Lemma, the set $\mathbf{\Delta}_1$ can actually be **any** diagonal augmentation of real and complex block structures.

This lemma implies that certain robustness conditions can be verified by examining the unit circle rather than the unit disc. For example, suppose $\rho(A) < 1$ and define $G(z) := D + C(zI - A)^{-1}B$ for $z \in \mathbf{C}$. Define

$$M := \begin{bmatrix} D & C \\ B & A \end{bmatrix}.$$

Then $F_l(M, \frac{1}{z}I_n) = G(z)$. Because of Lemma 6.2, we can write

$$\sup_{|z|\geq 1} \mu_1\left(G(z)\right) = \max_{|z|=1} \mu_1\left(G(z)\right). \tag{6.3}$$

These results can be applied to continuous-time systems by the bilinear transform equivalence of the open left-half-plane and the open unit disc. However, extrapolating these ideas to right-half-plane/imaginary axis results must be done with care. Consider G, a real-rational function, analytic in the closed right-half plane. In particular, let

$$G(s) := \frac{2s+1}{s+1}.$$

Let $\mathbf{\Delta} := \mathbf{R}$. Note that

$$\begin{aligned} \sup_{\omega\in\mathbf{R}} \mu_{\mathbf{\Delta}}\left(G(j\omega)\right) &= 1, \\ \sup_{\mathrm{Re}(s)\geq 0} \mu_{\mathbf{\Delta}}\left(G\left(s\right)\right) &= 2, \\ \mu_{\mathbf{\Delta}}\left(\lim_{s\to\infty} G(s)\right) &= 2. \end{aligned}$$

Hence, the supremum over the imaginary axis is not equal to the supremum over the right-half plane. This is due to the discontinuity at $s = \infty$. This problem does not arise in the discrete-time system. Indeed, in discrete time, i.e., for $s = \frac{1}{2}\frac{z-1}{z+1}$ we have

$$G(z) = \frac{4z}{3z+1}.$$

The only values of z, with $\mid z \mid = 1$, for which $\mu_1\left(G(z)\right)$ is nonzero are $z = 1$ and $z = -1$, and

$$\max_{|z|=1} \mu_1\left(G(z)\right) = 2.$$

The following theorem is the analog of (6.3) for continuous-time systems.

Theorem 6.4 *Suppose $P(s)$ is a $m \times m$ real-rational, proper transfer function matrix, which is analytic in the closed right-half-plane. Define $P_\infty := \lim_{s\to\infty} P(s)$. Let $\mathbf{\Delta}$ be any diagonal augmentation of the real and complex block structures defined in (6.2) and (4.1). Then*

$$\max\left\{\sup_{\mathrm{Re}(s)\geq 0} \mu_{\mathbf{\Delta}}\left(P(s)\right)\ ,\ \mu_{\mathbf{\Delta}}\left(P_\infty\right)\right\} = \max\left\{\sup_{\omega\in\mathbf{R}} \mu_{\mathbf{\Delta}}\left(P\left(j\omega\right)\right)\ ,\ \mu_{\mathbf{\Delta}}\left(P_\infty\right)\right\}.$$

Finally, for a certain class of real block structures, the function $\mu_{\mathbf{\Delta}}(\cdot)$ is continuous at all real matrices, M.

Lemma 6.5 *Let* $\mathbf{\Delta}$ *be a real block structure defined as*

$$\mathbf{\Delta} := \left\{ \Delta \in \mathbf{R}^{n \times n} : \Delta = \text{diag}[\delta_1, \ldots, \delta_n],\ \ \delta_i \in \mathbf{R} \right\}. \tag{6.4}$$

Then, for any $M \in \mathbf{R}^{n \times n}$, $\mu_{\mathbf{\Delta}} : \mathbf{R}^{n \times n} \to \mathbf{R}$ *is continuous at* M.

Proof: Without loss in generality, assume $\mu_{\mathbf{\Delta}}(M) \neq 0$. Define a function $f : \mathbf{R}^n \times \mathbf{R}^{n \times n} \to \mathbf{R}$ as

$$f(\delta_1, \ldots, \delta_n, M) := \det(I - M\Delta).$$

We are interested in finding a $\Delta \in \mathbf{\Delta}$ of minimum norm such that $\det(I - M\Delta) = 0$. Suppose $\bar{\Delta} := \text{diag}[\bar{\delta}_1, \ldots, \bar{\delta}_n]$ is such a solution, and consequently $f(\bar{\delta}_1, \ldots, \bar{\delta}_n, M) = 0$. Let

$$f(\delta_1, \bar{\delta}_2, \ldots, \bar{\delta}_n, M) = A_1 + B_1\delta_1 =: p_1(\delta_1, M)$$

where A_1 and B_1 are function of $\bar{\delta}_2, \ldots, \bar{\delta}_n$ and M. Clearly $p_1(\bar{\delta}_1) = 0$. If $B_1 = 0$ then $\bar{\Delta}_1 := \text{diag}[0, \bar{\delta}_2, \ldots, \bar{\delta}_n]$ is also a minimum norm solution. Continuing this way we obtain

$$f(0, \ldots, 0, \delta_r, \bar{\delta}_{r+1}, \ldots, \bar{\delta}_n, M) = A_r + B_r\delta_r =: p_r(\delta_r, M)$$

where $B_i = 0$ for $i = 1, \ldots, r-1$. Now there is an $r \in \{1, \ldots, n\}$ such that $B_r \neq 0$ (this is true because for $r = n$, we obtain $f(0, \ldots, 0, \delta_n) = 1 - M_{nn}\delta_n =: p_n(\delta_n, M)$ and $p_n(\bar{\delta}_n, M) = 0$, so $B_n = M_{nn} \neq 0$).

So, p_r is a nontrivial affine function in the real variable δ_r, and A_r and B_r are real-valued functions that depend continuously on M. Therefore, given $\epsilon > 0$, there exists a $\gamma > 0$ such that for all real matrices X with $\|X - M\| < \gamma$, there exists a real number $\tilde{\delta}_r$ satisfying

$$p_r\left(\tilde{\delta}_r, X\right) = 0 \qquad \text{and} \qquad \left|\bar{\delta}_r - \tilde{\delta}_r\right| < \epsilon$$

Hence, $\mu_{\mathbf{\Delta}}$ is continuous. ∎

Remark 6.6 The above result readily extends to the case where $\mathbf{\Delta}$ has full real blocks in addition to scalar reals. Specifically, suppose

$$\mathbf{\Delta} := \left\{ \Delta : \Delta = \text{diag}[\delta_1, \ldots, \delta_r, \Delta_1, \ldots, \Delta_k],\ \ \delta_i \in \mathbf{R}, \Delta_i \in \mathbf{R}^{n_i \times n_i} \right\} \tag{6.5}$$

As above, suppose $\bar{\Delta}$ is a minimum norm solution for $\det(I - M\Delta) = 0$. Let

$$\bar{\Delta} = \bar{U}\ \text{diag}[\bar{\delta}_1, \ldots, \bar{\delta}_n]\ \bar{V}^T$$

be a singular value decomposition of $\bar{\Delta}$. Now, apply the results of the previous lemma to the polynomial

$$\begin{aligned} p(\delta, M) &:= \det\left(I - M\bar{U}\text{diag}[\delta_1, \ldots, \delta_n]\bar{V}^T\right) \\ &= \det\left(I - \bar{V}^T M\bar{U}\text{diag}[\delta_1, \ldots, \delta_n]\right) \end{aligned}$$

to obtain continuity.

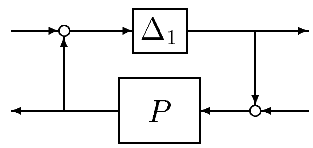

Figure 3: Linear Fractional Uncertainty Model

Remark 6.7 The above result is **not** true if $\mathbf{\Delta}$ contains repeated scalar blocks. For example, suppose $\mathbf{\Delta} := \{\delta I : \delta \in \mathbf{R}\}$, so that $\mu_{\mathbf{\Delta}}(M) = \rho_R(M)$. This quantity need not vary continuously with M. In particular, let $\mathbf{\Delta} := \{\delta I_2 : \delta \in \mathbf{R}\}$, and

$$M_\epsilon := \begin{bmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{bmatrix}.$$

For $\epsilon = 0$, we have $\mu_{\mathbf{\Delta}}(M_\epsilon) = 1$. However, for any $\epsilon \neq 0$, $\mu_{\mathbf{\Delta}}(M_\epsilon) = 0$.

Remark 6.8 Consider the setup as in Theorem 6.4, and $\mathbf{\Delta}$ is as in (6.5). Then,

$$\sup_{\mathrm{Re}(s) \geq 0} \mu_{\mathbf{\Delta}}(P(s)) = \max \left\{ \sup_{\omega \in \mathbf{R}} \mu_{\mathbf{\Delta}}(P(j\omega)) \ , \ \mu_{\mathbf{\Delta}}(P_\infty) \right\}.$$

This follows from the fact that real-rational transfer function matrices are real-valued for real values of s.

7 Robustness Margins

Consider a linear system with linear fractional uncertainty, as shown in Figure 3. Here $P(s)$ is a stable, proper, rational, $n \times n$ transfer function and the linear fractional perturbation is the Δ_1 element, which lies in some prespecified block structure $\mathbf{\Delta}_1 \subset \mathbf{C}^{n \times n}$. Suppose that $\mathbf{\Delta}_2$ is a **complex** block structure and $\mathbf{\Delta}_1 = \mathbf{R}^{n \times n} \bigcap \mathbf{\Delta}_2$. Let $\mathbf{\Delta}$ be the diagonal augmentation of these two structures.

Based on results obtained in the previous sections, consider two related problems:

- **Problem 1:** μ calculation for Real uncertainty,

$$\|P\|_{\mathbf{\Delta}_1} = \max \left\{ \sup_{\omega \in \mathbf{R}} \mu_1(P(j\omega)), \mu_1(P(\infty)) \right\} =: \frac{1}{R_1(P)}. \tag{7.1}$$

  By Theorem 3.1, for any $\beta > 0$, the closed-loop interconnection in Figure 3 is well-posed and stable for all $\Delta_1 \in \mathbf{\Delta}_1$ with $\bar{\sigma}(\Delta_1) \leq \beta$ if and only if $\beta < R_1(P)$.

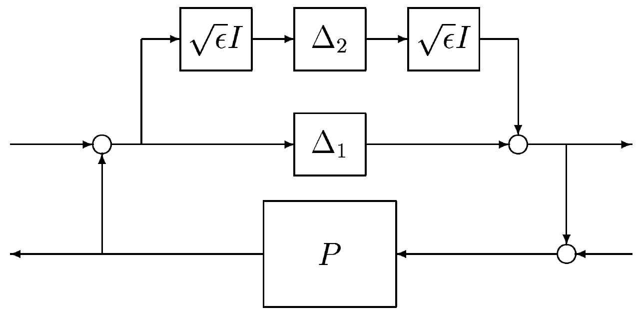

Figure 4: Problem 2 Uncertainty Model

- **Problem 2:** μ calculation for Real+Complex uncertainty. For $\epsilon > 0$ define $M(s)$ as

$$M(s) := \begin{bmatrix} P(s) & \sqrt{\epsilon}P(s) \\ \sqrt{\epsilon}P(s) & \epsilon P(s) \end{bmatrix}$$

and the μ calculation for Real+Complex uncertainty,

$$\begin{aligned} \|M\|_{\boldsymbol{\Delta}} &= \max\left\{\sup_{\omega\in\mathbf{R}} \mu_{\boldsymbol{\Delta}}\left(M(j\omega)\right), \mu_{\boldsymbol{\Delta}}\left(M(\infty)\right)\right\} \\ &= \sup_{\omega\in\mathbf{R}} \mu_{\boldsymbol{\Delta}}\left(M(j\omega)\right) \\ &=: \frac{1}{R_{2\epsilon}(P)}. \end{aligned} \tag{7.2}$$

The interconnection associated with Problem 2 is shown in Figure 4. Note that this is similar to the interconnection associated with Problem 1. However, now the linear fractional perturbation to P is $\Delta_1 + \epsilon\Delta_2$, where Δ_2 has the same structure as Δ_1, but may take on complex values. The value $R_{2\epsilon}(P)$ has the interpretation that for any $\beta > 0$, this closed-loop interconnection is well-posed and stable for all $\Delta_i \in \boldsymbol{\Delta}_i$ with $\bar{\sigma}(\Delta_i) \leq \beta$ if and only if $\beta < R_{2\epsilon}(P)$.

Note that Problem 1 involves purely real uncertainty, while the Problem 2 pertains to "nearly real" uncertainty. In the case of a scalar uncertainty block, it is easy to give a pictorial representation of the "unit" perturbations involved. In Problem 1, it is the interval $[-1, 1]$, while in Problem 2 it is the setwise summation of the real interval, and the complex disc of radius ϵ, centered at the origin. For each problem, these are shown Figure 5.

For these two problems, Theorem 7.1 (upcoming) confirms the following facts:

1. The margin $R_{2\epsilon}(P)$ computed in Problem 2 will be a continuous function of the problem data (ie. $P(s)$, when equipped with the $\|\cdot\|_\infty$ norm).

Problem 1, Unit Ball Uncertainty

Problem 2, "Unit Ball" Uncertainty

Figure 5: Uncertainty Sets for Problems 1 and 2

2. In Problem 2, the structured singular value $\mu_{\boldsymbol{\Delta}}$ will be continuous, so that the frequency domain maximization calculation of $\|M\|_{\boldsymbol{\Delta}}$ via gridding is possible (as the grid gets finer, the calculated maximum approaches the actual maximum).

3. Under the assumption of exact computation, and exact numerical representation, the margin $R_{2\epsilon}(P)$ approaches the margin $R_1(P)$ as ϵ gets arbitrarily small.

4. If $R_{2\epsilon}(P)$ is significantly less than $R_1(P)$, then **from an engineering point of view** the uncertainty model probably needs more attention. This is because in most engineering models, certain assumptions are made which ignore the dynamics of individual components. Without additional study though, the Problem 2 margin will always be on the conservative side of the Problem 1 margin.

Theorem 7.1 *Suppose $P(s)$ is a stable, proper, rational $n \times n$ transfer matrix. Define the set $\mathcal{G}$ as follows*

$$\mathcal{G} := \{P(j\omega) : \omega \in [0, \infty)\} \quad \cup \quad \{P_\infty\}. \tag{7.3}$$

Suppose that $\boldsymbol{\Delta}_2 \subset \mathbf{C}^{n\times n}$ is a complex block structure, and the real block structure $\boldsymbol{\Delta}_1 \subset \mathbf{R}^{n\times n}$ is related to $\boldsymbol{\Delta}_2$ by

$$\boldsymbol{\Delta}_1 = \mathbf{R}^{n\times n} \bigcap \boldsymbol{\Delta}_2.$$

Let $\boldsymbol{\Delta}$ be the diagonal augmentation of these two sets,

$$\boldsymbol{\Delta} := \{\mathrm{diag}[\Delta_1, \Delta_2] : \Delta_1 \in \boldsymbol{\Delta}_1, \Delta_2 \in \boldsymbol{\Delta}_2\}.$$

For any $G \in \mathcal{G}$ we define

$$M(\epsilon, G) := \begin{bmatrix} G & \sqrt{\epsilon} G \\ \sqrt{\epsilon} G & \epsilon G \end{bmatrix},$$

and the function

$$f(\epsilon, G) := \mu_{\boldsymbol{\Delta}}(M(\epsilon, G)).$$

The function $f:(0,\infty)\times\mathbf{C}^{n\times n}\rightarrow\mathbf{R}$ *is continuous. For* $\epsilon_1>\epsilon_2>0$*, we have* $f(\epsilon_1,G)\geq f(\epsilon_2,G)\geq\mu_1(G)$ *and*

$$\lim_{\epsilon\rightarrow 0} f(\epsilon,G)=\mu_1(G).$$

Further, given any $\delta>0$ *there is an* $\epsilon_*>0$ *such that for* ϵ *satisfying* $0<|\epsilon|\leq\epsilon_*$,

$$\|M(\epsilon,G)\|_{\mathbf{\Delta}}=\max_{G\in\mathcal{G}} f(\epsilon,G)<\|P\|_{\mathbf{\Delta}_1}+\delta.$$

Proof: Continuity of f follows from Lemma 5.8. Now, for any fixed $G\in\mathcal{G}$ we have a constant matrix problem, and by Lemma 2.5 we get

$$\lim_{\epsilon\rightarrow 0} f(\epsilon,G)=f(0,G)=\mu_1(G).$$

Thus, for a given $\delta>0$ there exists an $\epsilon_G>0$ such that for all $\epsilon\in[0,\epsilon_G]$ we have

$$f(\epsilon,G)<\mu_1(G)+\delta.$$

Since $f(G,\cdot)$ is continuous, there is a $r_G>0$ such that

$$f(\epsilon_G,\tilde{G})<\mu_1(G)+\delta,\qquad\forall\ \|G-\tilde{G}\|<r_G$$

Since $\mathcal{G}$ is compact we can choose a finite subcover $\{G_1,\ldots,G_N\}$, $\{\epsilon_{G_1},\ldots,\epsilon_{G_N}\}$, and $\{r_{G_1},\ldots,r_{G_N}\}$. Define $\epsilon_\star:=\min\{\epsilon_{G_1},\ldots,\epsilon_{G_N}\}>0$. Now choose any $G\in\mathcal{G}$ and any ϵ with $0\leq|\epsilon|<\epsilon_*$. Note that for some integer i with $1\leq i\leq N$, $\|G-G_i\|<r_{G_i}$. Hence,

$$\begin{aligned} f(\epsilon,G) &\leq f(\epsilon_*,G)\\ &\leq f(\epsilon_{G_i},G)\\ &< \mu_1(G_i)+\delta\\ &\leq \max_{G\in\mathcal{G}}\mu_1(G)+\delta\\ &= \|P\|_{\mathbf{\Delta}_1}+\delta \end{aligned}$$

which gives the desired result. ∎

Corollary 7.2 *Let* $g(\omega)$ *be a piecewise-constant function on* $\mathbf{R}$ *that takes on a finite set of values. As notation, let* g_∞ *be the value taken by* g *for arbitrarily large values of* ω*. Suppose that for all finite* ω*,* $\mu_1[P(\omega)]<g(\omega)$*. Then, there is an* $\epsilon_*>0$ *such that for all* $\epsilon\in(0,\epsilon_*)$ *and all* ω*,*

$$\begin{aligned} \mu_1[P(\omega)] &\leq \mu_{\mathbf{\Delta}}(M_\epsilon(\omega)) \leq g(\omega)\\ \mu_1[P_\infty] &\leq \mu_{\mathbf{\Delta}}(M_\epsilon(\infty)) \leq g_\infty. \end{aligned}$$

Proof: The function g has discontinuities at a finite number of points, say $\omega_1 < \omega_2 < \cdots < \omega_N$. Apply Theorem 7.1 locally to the compact sets

$$\begin{array}{lcl} \mathcal{G}_\infty & := & \{P(j\omega) : \omega \leq \omega_1 \text{ or } \omega \geq \omega_N\} \bigcup P_\infty \\ \mathcal{G}_1 & := & \{P(j\omega) : \omega_1 \leq \omega \leq \omega_2\} \\ \mathcal{G}_2 & := & \{P(j\omega) : \omega_2 \leq \omega \leq \omega_3\} \\ \vdots & \vdots & \vdots \\ \mathcal{G}_{N-1} & := & \{P(j\omega) : \omega_{N-1} \leq \omega \leq \omega_N\} \end{array}$$

to yield the bounds. ∎

Reiterating the consequences of these results, the margin $R_{2\epsilon}$ is, in general, more conservative than the original real margin R_1 (ie., $R_{2\epsilon} \leq R_1$), since it corresponds to perturbations lying in thin complex sets, rather than the real interval. The margin $R_{2\epsilon}$ is equivalent to a complexified structure singular value, $\mu_{\mathbf{\Delta}}$, via a frequency domain maximization, equation (7.2). The complexified structured singular value (and hence the margin $R_{2\epsilon}$) has certain continuity and convergence properties discussed below:

1. The function
$$f(\epsilon, M) := \mu_{\mathbf{\Delta}} \left(\begin{bmatrix} M & \sqrt{\epsilon} M \\ \sqrt{\epsilon} M & \epsilon M \end{bmatrix} \right)$$
is continuous when viewed as a function on the space $\mathbf{C}^{n \times m} \times (0, \infty)$.

2. Generalizing (1), the peak across frequency of the complexified structured singular value is continuous on the space of rational transfer matrices that have no poles in the closed right half plane, using the $\mathcal{H}_\infty$ norm on that space; hence if $\{P_i\}_{i=1}^\infty$ is a sequence of proper, stable transfer matrices, and P is also a proper, stable transfer matrix, and
$$\lim_{i \to \infty} \|P_i - P\|_\infty = 0,$$
then
$$\lim_{i \to \infty} R_{2\epsilon}(P_i) = R_{2\epsilon}(P).$$

3. For any fixed matrix M,
$$\lim_{\epsilon \to 0} f(\epsilon, M) = f(0, M) = \mu_1(M).$$

4. Generalizing (3), for any stable, rational transfer matrix, $P(s)$,
$$\lim_{\epsilon \to 0} R_{2\epsilon}(P) = R_1(P).$$
The convergence at each frequency is made precise in Corollary 7.2.

5. For a fixed value of $\epsilon > 0$, continuity of the margin $R_{2\epsilon}$ is guaranteed, and the interpretation of the relation between $R_{2\epsilon}(P)$ and $R_1(P)$ is:

   (a) the margin for real perturbations is larger than the margin for the complexified perturbations, $R_1(P) \geq R_{2\epsilon}(P)$.

   (b) If ϵ were allowed to change, and in fact converge to zero, then one would obtain, in the limit, $R_1(P)$ from $R_{2\epsilon}(P)$ (the complexified perturbation margin). **But**, it is never clear how small to actually take ϵ to get these quantities close, so we interpret the differences as follows:

      i. It may be that $R_1(P) \approx R_{2\epsilon}(P)$, (without computing both, this is impossible to **know**, but it may be true), in which case we are content to accept the answer from the complexified margin calculation, or...

      ii. It may be that $R_{2\epsilon}(P) << R_1(P)$ (also this is impossible to know, but it may be true). In this case, the actual robustness margin is very sensitive to the perturbation model, and it makes a big difference whether the perturbation is assumed to lie in a real interval, or whether the perturbation is assumed to lie in a very thin complex oval. At this point, **engineering judgement** suggests that we abandon the real-perturbation margin, $R_1(P)$, and accept the complexified margin $R_{2\epsilon}(P)$ as the answer, since our initial model that the perturbation is unequivocably real is probably not reasonable, given that all physical devices have some phase characteristics (to put it another way, if the margin for real perturbations is **completely different** than the margin for perturbations in a very thin complex oval, in an engineering problem, which would be more trustworthy?).

   Hence, in either case we accept the complexified margin $R_{2\epsilon}(P)$ as the answer. However, it is not claimed that for a fixed value of ϵ this will necessarily be close to $R_1(P)$. Also, if $\{M_i\}_{i=0}^{\infty}$ is a sequence of matrices with $M_i \to M$, and $\{\epsilon_i\}_{i=0}^{\infty}$ is a sequence of nonnegative real numbers, with $\epsilon_i \to 0$, then, in general,

   $$\lim_{i\to\infty} f(\epsilon_i, M_i) \neq f(0, M).$$

   Similarly, if $\{P_i\}_{i=0}^{\infty}$ is a sequence of stable rational matrices, converging (in $\mathcal{H}_\infty$) to a stable rational matrix P, then

   $$\lim_{i\to\infty} R_{2\epsilon_i}(P_i) \neq R_1(P).$$

It is important to note that we do not propose the complexified margin $R_{2\epsilon}(P)$ as a means to calculating the real-margin $R_1(P)$ by computing a sequence of robustness margins (each with guaranteed continuity properties) and using the limiting value as the margin. The reason is that the limiting value, which is the real-margin, **is not a continuous function of the data**. Hence, while one would experience no discontinuity problems in each of the complexified computations, estimating the limit would be fruitless, since **it** is a discontinuous function of the problem data.

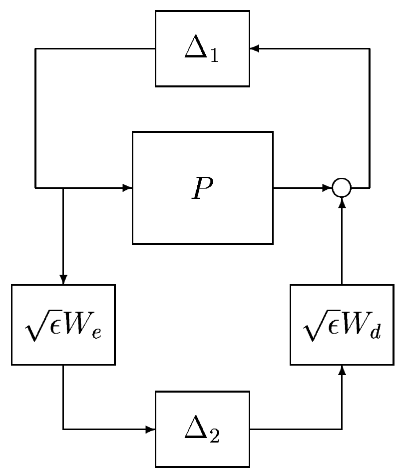

Figure 6: Problem 3 Uncertainty Model

Instead, for engineering problems where one would *like* to model the uncertainty as real, we propose a **new margin**, based on thin ovals in the complex plane. This margin enjoys many continuity properties, and, if the problem data is held constant, converges to the real-margin as the thickness of the complex sets converges to zero. We propose that this complexified margin is simply accepted as the appropriate margin for stability with respect to real perturbations.

The same convergence and continuity properties which relate $R_{2\epsilon}$ to R_1 can be derived for the regularization methods given in Lemmas 5.7 and 5.10. Specifically, define two additional problems

- **Problem 3, μ calculation for a Complex Full Block of additive uncertainty:** Suppose the original stability robustness problem (Problem 1) is modified to include weighted, **unmodelled dynamics**, in the form of a full, complex block of additive uncertainty, Δ_2, weighted with stable, minimum phase, weighting matrices $W_d(s)$ and $W_e(s)$. Specifically, let $\epsilon > 0$, and define $M(s)$ as

  $$M(s) := \begin{bmatrix} P(s) & \sqrt{\epsilon}W_e(s) \\ \sqrt{\epsilon}W_d(s) & 0 \end{bmatrix}.$$

  Let $\mathbf{\Delta}_2 := \mathbf{C}^{n \times n}$. As usual, define $\mathbf{\Delta}$ to be the diagonal augmentation of the two sets $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$. The stability radius in this augmented problem is

  $$\frac{1}{R_{3\epsilon}} := \|M\|_{\mathbf{\Delta}}$$

  The associated interconnection is shown in Figure 6.

  By Lemma 5.7, the robustness margin with respect to the original uncertainty and the additional complex uncertainty will be a continuous function of the problem data, P, ϵ, W_d and W_e (on the space of proper, rational, stable transfer matrices, with the $\|\cdot\|_\infty$ norm). Furthermore, as $\epsilon \to 0$, the computed margin $R_{3\epsilon}$ approaches R_1.

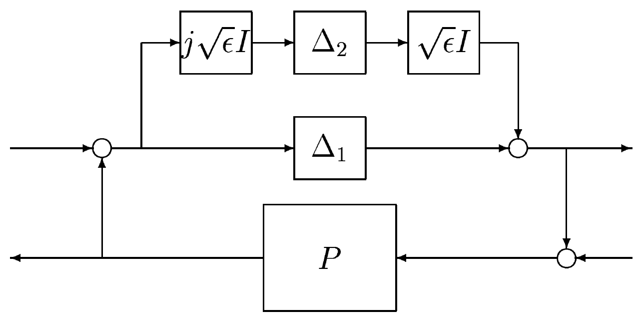

Figure 7: Problem 4 Uncertainty Model

- **Problem 4, Real+Imaginary uncertainty:** For any $\epsilon > 0$, define

$$M(s) := \begin{bmatrix} P(s) & \sqrt{\epsilon}P(s) \\ j\sqrt{\epsilon}P(s) & j\epsilon P(s) \end{bmatrix}$$

Let $\boldsymbol{\Delta}_2 := \boldsymbol{\Delta}_1$ ($\boldsymbol{\Delta}_1$ is the original given real block structure). Define $\boldsymbol{\Delta}$ as the diagonal augmentation of these two structures. The stability margin in this augmented problem is given by

$$\frac{1}{R_{4\epsilon}} := \|M\|_{\boldsymbol{\Delta}}$$

This test applies to the interconnection shown in Figure 7, where the linear fractional perturbation to P (originally Δ_1) is replaced with $\Delta_1 + j\epsilon\Delta_2$, and Δ_2 is a real matrix, with the exact same structure as Δ_1. Again, for any $\epsilon \neq 0$, $R_{4\epsilon} \leq R_1$, and $R_{4\epsilon} \to R_1$ as $\epsilon \to 0$.

8 Example

Consider the example presented in [3] which involves the following polynomial:

$$p(s, \delta_1, \delta_2, q) := s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4, \tag{8.4}$$

where the coefficients a_i depend on perturbations $\delta_i \in \mathbf{R}$, and a variable $q \in \mathbf{R}$. Specifically,

$$\begin{aligned} a_1 &= 20 - 20\delta_2, & a_2 &= 44 + 2a + 10\delta_1 - 40\delta_2, \\ a_3 &= 20 + 8a + 20a\delta_1 - 20\delta_2, & a_4 &= a(5a - 4q) + 10a(a - q)\delta_1, \end{aligned}$$

and $a := 3 + 2\sqrt{2}$ is a constant. The δ's are interpreted as perturbations, while q is interpreted as a parameter which continuously varies the problem data. To compute the stability margin

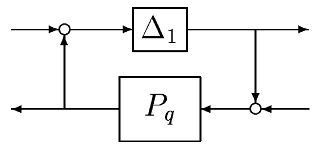

Figure 8: Perturbed closed-loop system

using the μ framework we define the following (q-dependent) nominal state-space data:

$$A_q := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a(5a-4q) & -(20+8a) & -(44+2a) & -20 \end{bmatrix}, \qquad B_q := -\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

and

$$C_q := -\begin{bmatrix} 0 & -20 & -40 & -20 \\ 10a(a-q) & 20a & 10 & 0 \end{bmatrix}, \qquad D_q := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Define the associated real block structure $\mathbf{\Delta}_1 := \{\mathrm{diag}\,[\delta_1, \delta_2] : \delta_i \in \mathbf{R}\}$. Note that the polynomial p is the characteristic polynomial of the system

$$\dot{x} = \left[A_q + B_q\Delta_1 \left(I - D_q\Delta_1\right)^{-1} C_q\right] x. \tag{8.5}$$

Define

$$P_q(s) := D_q + C_q(sI - A_q)^{-1}B_q =: \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$$

(in writing P_{ij}, we have suppressed the q dependence). This is the transfer function that the perturbation Δ_1 "sees" in system defined in (8.5). Hence, for fixed values of q, δ_1 and δ_2, the polynomial $p(s, \delta_1, \delta_2, q)$ has all of its roots in the open-left-half plane if and only if the closed-loop system shown in Figure 8 is internally stable.

The actual state-space data gives that for all ω, and for all q

$$\begin{aligned} P_{11}(j\omega, q) &= P_{12}(j\omega, q) \\ P_{21}(j\omega, q) &= P_{22}(j\omega, q). \end{aligned}$$

Hence, define real, scalar functions of frequency $x_i(\omega, q)$ and $y_i(\omega, q)$ by $P_{11}(j\omega, q) = x_1(\omega, q) + jx_2(\omega, q)$ and $P_{21}(j\omega, q) = y_1(\omega, q) + jy_2(\omega, q)$. Simple manipulations show that for all $\Delta_1 \in \mathbf{\Delta}_1$, $\det\left(I - P_q(j\omega)\Delta_1\right) = 0$ if and only if

$$\begin{bmatrix} x_1(\omega, q) & y_1(\omega, q) \\ x_2(\omega, q) & y_2(\omega, q) \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{8.6}$$

Hence

$$\frac{1}{\mu_1\left(P_q\left(j\omega\right)\right)} = \min\left\{\max\left(\left|\delta_1\right|, \left|\delta_2\right|\right) : \text{ subject to condition 8.6}\right\}.$$

For both zero and nonzero values of q, this particular linear program can be solved exactly (in [3], the solution is derived using Routh-Hurwitz), giving

$$q = 0: \qquad \frac{1}{\mu_1\left(P_{[q=0]}(j\omega)\right)} = \begin{cases} \max\left(\left|\frac{\omega^2-5}{10}\right|, \left|1-\frac{a}{10}\right|\right), & \omega \neq \sqrt{a},\ \omega \neq 0, \\ \frac{1}{5}(7-a), & \omega = \sqrt{a} \\ \frac{1}{2} & \omega = 0 \end{cases}$$

$$q \neq 0: \qquad \frac{1}{\mu_1\left(P_{[q\neq 0]}(j\omega)\right)} = \begin{cases} \max\left(E_q(\omega), G_q\left(\omega\right)\right) & \omega \neq 0, \\ \frac{5a-4q}{10(a-q)} & \omega = 0 \end{cases}$$

where

$$E_q(\omega) := \left|\frac{(\omega^2-a)^2(\omega^2-1)}{10\Gamma_q(\omega)} - 0.4\right|,$$

$$G_q(\omega) := \left|\frac{a(\omega^2-a)^2}{10\Gamma_q(\omega)} - 1\right|,$$

with

$$\Gamma_q(\omega) := \omega^4 - a(2-q)\omega^2 + a(a-q).$$

These are shown in Figure 9. The discontinuous curve corresponds to $q = 0$, while the other curves correspond to decreasing values of q from 10^{-1} to 10^{-6}. Note that as $q \to 0$, the curves with $q \neq 0$ do **not** approach the $q = 0$ curve.

Next, consider the robustness formulation described by **Problem 4** in Section 7. Effectively, this recasts the original **real** perturbation problem as a real+imaginary perturbation problem. For a fixed value of $\epsilon > 0$, define

$$M\left(\epsilon, P_q(j\omega)\right) := \begin{bmatrix} P_q(j\omega) & \sqrt{\epsilon}P_q(j\omega) \\ j\sqrt{\epsilon}P_q(j\omega) & \epsilon P_q(j\omega) \end{bmatrix}$$

and

$$\mathbf{\Delta} := \left\{\text{diag}\left[\delta_1, \delta_2, \delta_3, \delta_4\right] : \delta_i \in \mathbf{R}\right\}.$$

For any $\Delta \in \mathbf{\Delta}$, we have $\det\left(I - M\left(\epsilon, P_q(j\omega)\right)\Delta\right) = 0$ if and only if

$$\det\left(I - P_q(j\omega)\begin{bmatrix} \delta_1 + \epsilon j\delta_3 & 0 \\ 0 & \delta_2 + \epsilon j\delta_4 \end{bmatrix}\right) = 0.$$

As before, this holds if and only if

$$\begin{bmatrix} x_1(\omega, q) & -\epsilon x_2(\omega, q) & y_1(\omega, q) & -\epsilon y_2(\omega, q) \\ x_2(\omega, q) & \epsilon x_1(\omega, q) & y_2(\omega, q) & \epsilon y_1(\omega, q) \end{bmatrix}\begin{bmatrix} \delta_1 \\ \delta_3 \\ \delta_2 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{8.7}$$

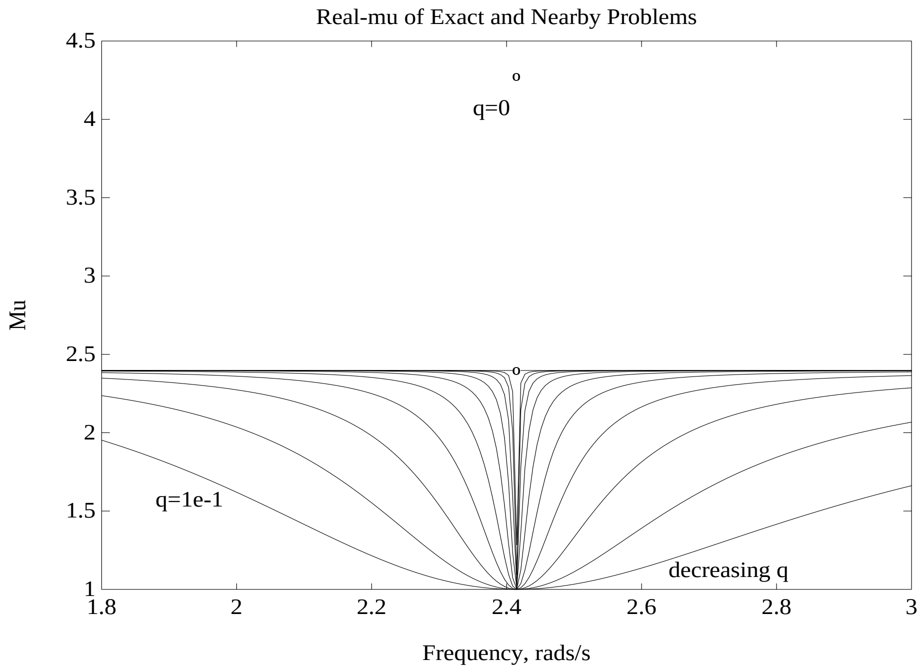

Figure 9: $\mu_1 \left[P_q(j\omega)\right]$ for the Exact Problem $(q = 0)$ and for several values of q in $(10^{-6}, 10^{-1})$.

Hence,

$$\frac{1}{\mu_{\mathbf{\Delta}}\left(M(\epsilon, P_q(j\omega))\right)} = \min\left\{\max\left(|\delta_1|, |\delta_2|, |\delta_3|, |\delta_4|\right) : \text{ subject to condition 8.7}\right\}.$$

We have solved this linear program for the following cases:

- In Figure 10, we have fixed the value of ϵ to be 10^{-5} and show several plots of $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_q(j\omega)\right)\right)$, with q taking on values between 0 and 10^{-1}. Note the continuity of $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_q(j\omega)\right)\right)$ as a function of the problem data (q). Contrast this with Figure 9, which is a similar plot, but for $\mu_1\left(P_q(j\omega)\right)$, or equivalently, for $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_q(j\omega)\right)\right)$ with $\epsilon = 0$.
- In Figure 11, the plots are $\mu_1\left(P_{[q=0]}(j\omega)\right)$ and $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_{[q=0]}(j\omega)\right)\right)$, for 5 values of ϵ logarithmically spaced between 10^{-3} and 10^{-5}. Note the following:
  - For all ω and all $\epsilon > 0$, $\mu_1\left(P_{[q=0]}(j\omega)\right) \leq \mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_{[q=0]}(j\omega)\right)\right)$.
  - $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_{[q=0]}(j\omega)\right)\right)$ is a continuous function of ω .
  - As $\epsilon \to 0$, $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_{[q=0]}(j\omega)\right)\right)$ converges to $\mu_1\left(P_{[q=0]}(j\omega)\right)$, as described in Theorem 7.1 and Corollary 7.2.

- Finally, in Figure 12, we are computing $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_{[q=0.001]}(j\omega)\right)\right)$ for 5 values of ϵ logarithmically spaced between 10^{-3} and 10^{-5}, along with $\epsilon = 0$. We see that as $\epsilon \to 0$, $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_{[q=0.001]}(j\omega)\right)\right)$ converges to $\mu_1\left(P_{[q=0.001]}(j\omega)\right)$, as described in Theorem 7.1 and Corollary 7.2.

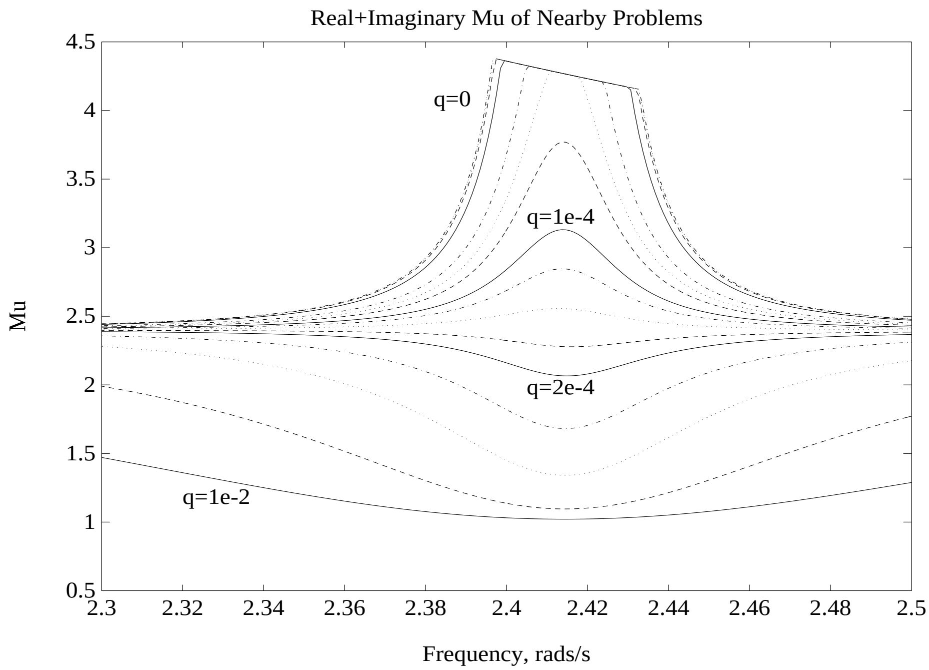

Figure 10: $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_q(j\omega)\right)\right)$ with $\epsilon = 10^{-5}$ and 14 values of q in $(0, 10^{-1})$.

After this paper was submitted, formulae for the rank-1 mixed-μ problem appeared, [5], [6]. The example discussed in this section involves a rank-1 matrix at each frequency. In our calculations, we used the fact that $P(j\omega)$ has rank equal to 1 to reformulate the μ calculation as a linear program. Thanks to the formulae of [5], this approach is unnecessary. We have recalculated the complexified structured singular value using these formulae, and at all frequency points, the new formula produced (on digital computer) the same answer as the solution given by the linear program.

9 Conclusions

Some continuity properties of the structured singular value have been derived, as well as three methods for regularizing robustness problems with real uncertainty. More research is needed in this general area, along with additional tests which can guarantee continuity without such restrictive uncertainty assumptions as in Lemmas 5.7, 5.8, and 5.10. Current work along those lines can be found in [17], who analyzes the discontinuity in robustness problems of the form

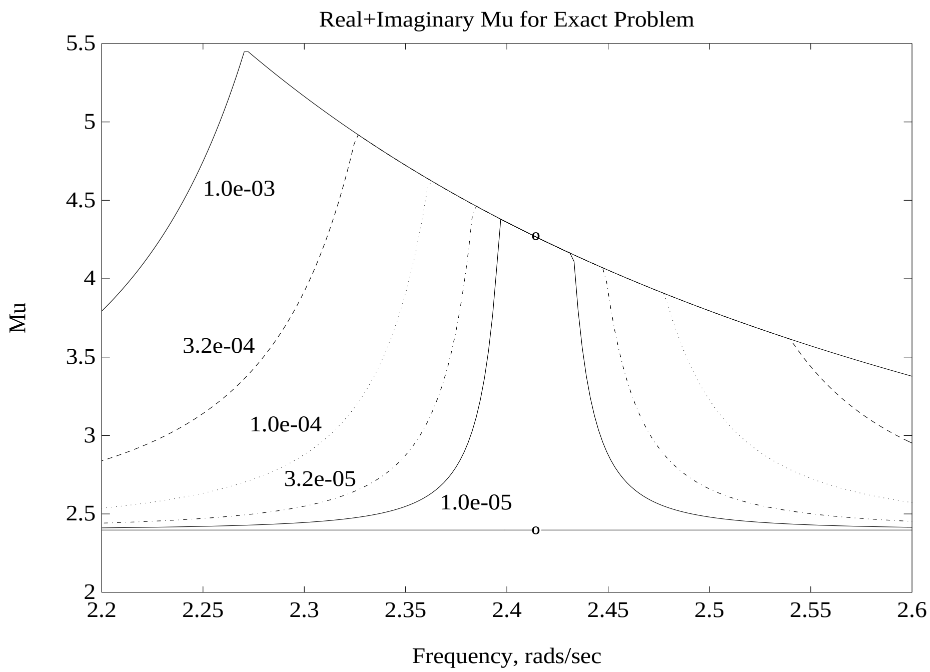

Figure 11: $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_q(j\omega)\right)\right)$ with $q = 0$ and 5 values of ϵ in $(10^{-5}, 10^{-3})$. ($\mu_1\left[P_{[q=0]}(j\omega)\right]$ is also shown for reference).

considered in [3]. Also, **condition number** estimates on the structured singular value are needed since it is expected that at points of discontinuity, the three augmented problems in Section 7 will still have very high condition numbers.

10 Acknowledgements

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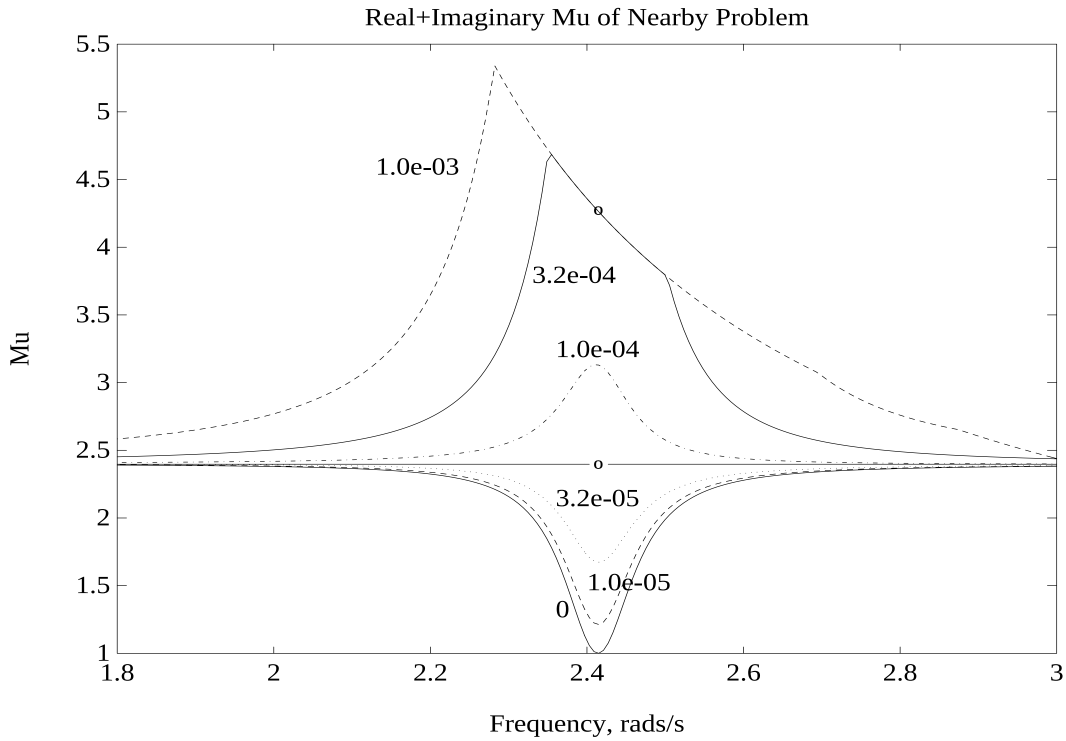

Figure 12: $\mu_{\mathbf{\Delta}}\left(M\left(\epsilon, P_q(j\omega)\right)\right)$ with $q = 10^{-3}$ and 5 values of ϵ in $(10^{-5}, 10^{-3})$, along with $\epsilon = 0$. $\left(\mu_1\left[P_{[q=0]}(j\omega)\right]\right.$ is also shown for reference).

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