Gain Scheduling via Linear Fractional Transformations

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Abstract

A linear, finite-dimensional plant, with rational state-space parameter dependence, is controlled using a parameter-dependent controller. The parameters are known to take on values in a unit ball, and are known in real-time. The goal of control is to stabilize the parameter-dependent closed-loop system, and provide disturbance/error attenuation as measured in induced $\mathbf{l}_2$ norms. The approach taken uses the optimally scaled small gain theorem, and solves the control synthesis problem by reformulating the existence conditions into a finite dimensional convex optimization.

1 Introduction

The recent developments in state-space $\mathcal{H}_\infty$ theory are almost exclusively expressed in terms of stabilizing solutions to a pair of Riccati equations satisfying a spectral radius coupling condition. Alternatively, it is possible to derive necessary and sufficient conditions in the form of 3 Affine Matrix Inequalities (AMI), [8], [22], [21], and [2]. The affine matrix inequalities represent convex constraints on the space of symmetric, positive definite matrices, and their feasability (or infeasability) determines the solvability (or unsolvability) of the sub-optimal $\mathcal{H}_\infty$ control problem. Here, we take the AMI approach, and solve a generalized $\mathcal{H}_\infty$ control problem, [12], [17], [13], where the plant **and** controller depend on many parameters (not just delays) in a linear fractional manner. When such a controller is implemented, the resulting closed-loop system has repeated linear fractional parameter dependence (since the parameters appear in both the plant an controller) as well as the usual exogenous disturbance and error signals. One technique to analyze the robust performance of such a system is to use the optimally scaled small-gain theorem. The scaling matrices are block-diagonal, constant matrices, whose dimensions are compatible with the repeated parameters.

Using the AMI approach, we are able to cast the existence of a parameter-dependent controller and a block-structured scaling matrix such that the scaled closed-loop $\mathcal{H}_\infty$ norm is less than 1 into the feasability of an AMI. It is also shown that for the sub-optimal problem, each linear fractional parameter must be repeated in the controller **no more than the number of times it appears in the plant**. This is entirely analogous to the well known fact that in sub-optimal $\mathcal{H}_\infty$ control, the controller state dimension can be taken to be equal to the plant state dimension.

The implications for gain-scheduling are obvious, since gain-scheduling conceptually involves a linear, parameter-dependent plant. The parameter-dependence can arise in a linear model, [23], or in a parametrized family of Jacobian linearizations, [24], or from exact linearization techniques, [24], [20]. This approach allows us to treat gain-scheduled controllers as a single entity, with the gain-scheduling being effected entirely using linear fractional transformations on the time-varying parameters.

The main motivation for our work lies in [11], [10], [23], [24], [25], and [20]. These discuss linear, parameter varying systems (LPVs) and their importance in gain-scheduling design. Specifically, [11] studies factorizations and realizations of systems over rings. This class of systems includes linear, time-invariant, parameter-dependent systems. In [10], control of time-invariant parameter-dependent systems is considered. The theorems are concerned with the parameter-dependent stabilization, observation and control, and concentrate on the parameter-dependence that is necessary in the controller's state-space entries. Gain-scheduling for linear, parameter-dependent systems is treated in [23]. They derive sufficient conditions for the existence of a parameter-dependent controller which guarantees stability for classes of time-varying parameters. Theoretical issues associated with controlling nonlinear systems using a gain-scheduling perspective are studied in [25]. Several heuristic rules-of-thumb about gain-scheduling are given theoretical interpretation and justification. In [24], a special class of nonlinear systems are introduced called "quasi-LPV." These types of nonlinear systems can be made to look like linear, parameter-dependent systems using a global diffeomorphism. Finally, [20] is a complete overview of the extended linearization approach to gain-scheduling.

The notation is standard. $\mathbf{F}$ denotes either the set of real or complex numbers. $\mathbf{C}^{n\times m}$ and $\mathbf{R}^{n\times m}$ are respectively the set of complex and real $n \times m$ matrices. For $M \in \mathbf{C}^{n\times n}$, the maximum singular value of M is denoted by $\bar{\sigma}(M)$. M^* is the complex conjugate transpose of M. For square matrices, $\rho(\cdot)$ is the spectral radius. If $M = M^*$, then all the eigenvalues of M are real, and the notation $\lambda_{\max}(M)$ is clear. Also for $M = M^*$, the notation $M > 0$ (< 0) indicates that M is positive (negative) definite, and for positive definite matrices, $M^{\frac{1}{2}}$ denotes the Hermitian square root. If X is any set, then $\mathcal{M}(X)$ will denote matrices whose elements are in X, where the dimensions of the matrices is clear from the context. For a rational, transfer function matrix $G(z)$, if all the poles of $G(z)$ are in the open unit disk, then

$$\|G\|_\infty := \sup_{\Omega\in[0,2\pi]} \bar{\sigma}\left[G\left(e^{j\Omega}\right)\right] = \sup_{|z|\geq 1} \bar{\sigma}\left[G(z)\right].$$

If $\{S_i\}_{i=1}^{i=N}$ is a collection of matrices, then $\mathbf{diag}(S_i)$ denotes the block diagonal augmentation of these matrices. For a 2×2 block-partitioned matrix M, and a matrix Q satisfying $\det(I - M_{22}Q) \neq 0$, the linear fractional transformation $\mathcal{F}_l(M,Q)$ is defined $\mathcal{F}_l(M,Q) := M_{11} + M_{12}Q(I - M_{22}Q)^{-1} M_{21}$. Finally, $\mathbf{l}_2$ is the space of vector-valued, square summable sequences (this paper considers a discrete-time problem, continuous-time systems would be treated similarly).

2 Problem Description: Plant; Controller; Closed-Loop structure

In this section, we define a special class of time-varying linear systems, called parameter-dependent LFT systems, [12], [24], [25], [17], [13], [2]. Suppose that P is 3-input, 3-output (each of the channels may itself be multivariable) time-invariant linear system, with internal state description given by

$$\begin{bmatrix} x_{k+1} \\ \alpha_k \\ e_k \\ y_k \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sp} & B_{1s} & B_{2s} \\ A_{ps} & A_{pp} & B_{1p} & B_{2p} \\ C_{1s} & C_{1p} & D_{11} & D_{12} \\ C_{2s} & C_{2p} & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x_k \\ \beta_k \\ d_k \\ u_k \end{bmatrix}$$

where the dimensions of the signals are $x_k \in \mathbf{R}^{n_o}, \alpha_k, \beta_k \in \mathbf{R}^{n}, e_k \in \mathbf{R}^{n_e}, d_k \in \mathbf{R}^{n_d}, y_k \in \mathbf{R}^{n_y}, u_k \in \mathbf{R}^{n_u}$, and the matrices are of compatible dimensions. We use P, which is time-invariant, to define a special class of **time-varying** linear systems. Let $\mathcal{N}$ be a f-tuple of positive integers, $\mathcal{N} = (n_1, n_2 \ldots, n_f)$, with $n := \sum_{i=1}^{f} n_i$. For any $\delta \in \mathbf{C}^f$, let $\mathcal{L}_{\mathcal{N}}[\delta]$ be the diagonal matrix

$$\mathcal{L}_{\mathcal{N}}[\delta] := \text{diag}\left[\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_f I_{n_f}\right] \in \mathbf{C}^{n\times n}$$

Consider a time-varying system which is the interconnection of $\mathcal{L}_{\mathcal{N}}[\delta(k)]$ with P as shown in Figure 1. The channels of inputs/outputs into P are respectively: linear fractional feedback channels (α, β), through which the time-varying δ parameters act, arranged in the block diagonal matrix $\mathcal{L}_{\mathcal{N}}[\delta]$; exogenous disturbances (d)/exogenous errors (e); control (u)/measurements (y). For any parameter sequence $\{\delta(k)\}_{k=0}^{\infty}$, the interconnection represents a time-varying linear system. The state equations are

$$\begin{bmatrix} x_{k+1} \\ e_k \\ y_k \end{bmatrix} = M_k \begin{bmatrix} x_k \\ d_k \\ u_k \end{bmatrix}$$

where M_k is given by

$$M_k = \begin{bmatrix} A_{ss} & B_{1s} & B_{2s} \\ C_{1s} & D_{11} & D_{12} \\ C_{2s} & D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} A_{sp} \\ C_{1p} \\ C_{2p} \end{bmatrix} \mathcal{L}_{\mathcal{N}}[\delta(k)] \left(I - A_{pp}\mathcal{L}_{\mathcal{N}}[\delta(k)]\right)^{-1} \begin{bmatrix} A_{ps} & B_{1p} & B_{2p} \end{bmatrix}.$$

Hence, the state-space entries are allowed to be arbitrary rational functions of the parameters $\delta_1, \ldots, \delta_f$. The typical assumption we make is that under operation of the plant, each of the parameters is time-varying, and satisfies $\sup_{k \geq 0} |\delta_i(k)| \leq 1$ for all $1 \leq i \leq f$. We also assume that A_{pp} is such that $(I - A_{pp}\mathcal{L}_{\mathcal{N}}[\delta])$ is invertible for all such matrices $\mathcal{L}_{\mathcal{N}}[\delta]$. This guarantees that for all parameter variations, the plant is indeed a causal operator from (d, u) to (e, y).

The controller for this parameter-dependent LFT plant will also be parameter-dependent, and will be restricted to have a similar structure – a linear fractional interconnection of a linear, time-invariant system with copies of the same δ parameters which affect the plant. The state space model for the controller is

$$\begin{bmatrix} \tilde{x}_{k+1} \\ u_k \\ \tilde{\alpha}_k \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ y_k \\ \tilde{\beta}_k \end{bmatrix} \quad , \quad \tilde{\beta}_k = \mathcal{L}_{\mathcal{R}}[\delta(k)]\, \tilde{\alpha}_k \tag{2.1}$$

where $\mathcal{R}$ is a f-tuple of nonnegative integers, $\mathcal{R} = (r_1, r_2, \ldots, r_f)$ which indicate how many copies of the plant's δ_i parameter that the LFT controller will depend on. We emphasize that at this point in the problem motivation, it is not clear what the state dimension of the controller, or how many copies of the plant's δ_i parameter that the LFT controller should use. Hence the controller is only restricted to be a finite-dimensional linear system with **arbitrary** rational dependence on the parameters. Later, when we precisely state the performance objective, we will **derive** (in theorem 6.3) the appropriate value for r_i. Supressing the controller's states, an input/output diagram of the controller structure is shown in Figure 2. The invertibility of of the matrix $I - \tilde{D}_{22}\mathcal{L}_{\mathcal{R}}[\delta(k)]$ is discussed in Section 6.

By interconnecting the parameter-dependent plant with the parameter-dependent controller, the closed-loop system appears in Figure 3. Note that the closed-loop system depends on $n_i + r_i$ copies of the parameter δ_i, hence define an operator $\mathcal{L}_{\mathcal{N}:\mathcal{R}} : \mathbf{C}^f \to \mathbf{C}^{(n+r)\times(n+r)}$ as

$$\mathcal{L}_{\mathcal{N}:\mathcal{R}}[\delta] := \operatorname{diag}\left[\delta_1 I_{n_1}, \ldots, \delta_f I_{n_f}, \delta_1 I_{r_1}, \ldots, \delta_f I_{r_f}\right] = \begin{bmatrix} \mathcal{L}_{\mathcal{N}}[\delta] & 0 \\ 0 & \mathcal{L}_{\mathcal{R}}[\delta] \end{bmatrix}.$$

Now, add two extra inputs and outputs to P, defining a new linear, time-invariant system $P_{\mathcal{R}}$,

$$P_{\mathcal{R}} := \begin{bmatrix} P_{11} & 0 & P_{12} & P_{13} & 0 \\ 0 & 0_r & 0 & 0 & I_r \\ P_{21} & 0 & P_{22} & P_{23} & 0 \\ P_{31} & 0 & P_{32} & P_{33} & 0 \\ 0 & I_r & 0 & 0 & 0 \end{bmatrix}$$

Then, the closed-loop system can be drawn as in Figure 4. Note that we have collected all of the time-varying parameters (both from the plant and controller) together, and the $P_{\mathcal{R}}$ and K are linear time-invariant systems. $P_{\mathcal{R}}$ is completely known, though the dimensions of the extra inputs/outputs (r) is not yet known. The system K is the LTI portion of the LFT controller. Drawn in this manner, the synthesis problem appears as a robust control problem, with a very special structure on the generalized plant, $P_{\mathcal{R}}$ and the "uncertainties" $\mathcal{L}_{\mathcal{N}:\mathcal{R}}$. The control objective is to pick a nonnegative integer vector $\mathcal{R}$, and design a controller K such that for all allowable parameter trajectories $\delta(k)$, the perturbed system shown in Figure 3 is stable and has a small induced norm from d to e. The induced $\mathbf{l}_2$ norm of the $d \rightarrow e$ channel, under time-varying $\delta_i(k)$, can be (conservatively) bounded using the optimally scaled small-gain theorem. The next section discusses the scaling matrices that are allowed, and makes precise the overall control synthesis objective. In Section 4, the problem is converted into a constant-matrix problem, using elementary state-space manipulations. In Section 5, some lemmas which are relevant to our constant matrix problem are stated. Finally, in Section 6, Theorem 6.3 casts the synthesis problem as a convex feasability problem, which is the main result of the paper.

3 Control Objective

Since the perturbations $\mathcal{L}_{\mathcal{N}}[\delta]$ have special repeated and block diagonal structure, it is possible to use inout/output similarity scalings to reduce the conservatism of the small-gain theorem. In particular, define the set $\mathbf{J}(\mathcal{N},\mathcal{R}) \subset \mathbf{R}^{(n+r)\times(n+r)}$

$$\mathbf{J}(\mathcal{N},\mathcal{R}) := \left\{ \begin{bmatrix} \mathbf{diag}(J_{11i}) & \mathbf{diag}(J_{12i}) \\ \mathbf{diag}(J_{21i}) & \mathbf{diag}(J_{22i}) \end{bmatrix} : \begin{bmatrix} J_{11i} & J_{12i} \\ J_{21i} & J_{22i} \end{bmatrix} \in \mathbf{R}^{(n_i+r_i)\times(n_i+r_i)}, \text{ invertible}, 1 \leq i \leq f \right\}$$

Note that for any $\delta \in \mathbf{C}^f$, and any $J \in \mathbf{J}(\mathcal{N},\mathcal{R})$, $J\mathcal{L}_{\mathcal{N}:\mathcal{R}}[\delta] = \mathcal{L}_{\mathcal{N}:\mathcal{R}}[\delta]J$. Equivalently, for any time-varying gains $\delta(k)$, and any $J \in \mathbf{J}(\mathcal{N},\mathcal{R})$, the equality $\mathcal{L}_{\mathcal{N}:\mathcal{R}}[\delta(k)] = J^{-1}\mathcal{L}_{\mathcal{N}:\mathcal{R}}[\delta(k)]J$ holds for all k. Using the small-gain theorem for linear systems with linear fractional uncertainties, we easily have the following lemma. This lemma uses the scaling set to bound the induced $\mathbf{l}_2$ performance of the parameter dependent closed-loop system, [5], [16].

Lemma 3.1 *Suppose P and $\mathcal{N}$ are defined as in section 2. Let $\mathcal{R}$ be a f-tuple of nonnegative integers. If there is a $J \in \mathbf{J}(\mathcal{N},\mathcal{R})$ and a stabilizing, finite-dimensional, linear, time-invariant K such that*

$$\left\| \begin{bmatrix} J & 0 \\ 0 & I_{n_e} \end{bmatrix} \mathcal{F}_l(P_{\mathcal{R}},K) \begin{bmatrix} J^{-1} & 0 \\ 0 & I_{n_d} \end{bmatrix} \right\|_\infty < 1 \tag{3.2}$$

then there is a $0 \leq \gamma < 1$, such that for all parameter sequences $\delta_i(\cdot)$ with $\sup_{k\geq 0}|\delta_i(k)| \leq 1$, the system in in Figure 4 is well-posed (for a given disturbance signal d, all signals $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}, e, y$ and u are unique, and each depends causally on δ and d) and internally exponentially stable, moreover for zero initial conditions, if $d \in \mathbf{l}_2$, then $\|e\|_2 \leq \gamma \|d\|_2$.

Remark 3.2 *If the plant P and controller K are restricted to be finite-dimensional, time-invariant linear systems, with state space entries in $\mathbf{R}$, then it is easy to show ([18], [16]) that allowing the scaling matrices in $\mathbf{J}(\mathcal{N},\mathcal{R})$ to take on complex values offers no advantage in reducing the scaled $\|\cdot\|_\infty$ norm.*

Based on lemma 3.1, we formulate the LFT control synthesis problem as:

Definition 3.3 (LFT Control Synthesis Problem:) Given P and $\mathcal{N}$ as defined in section 2. The *LFT control synthesis problem* associated with this open-loop data is **solvable** if there exists an f-tuple of nonnegative integers, $\mathcal{R}$, a matrix $J \in \mathbf{J}(\mathcal{N}, \mathcal{R})$ and a stabilizing, finite-dimensional, linear, time-invariant K such that

$$\left\| \begin{bmatrix} J & 0 \\ 0 & I_{n_e} \end{bmatrix} \mathcal{F}_l\left(P_{\mathcal{R}}, K\right) \begin{bmatrix} J^{-1} & 0 \\ 0 & I_{n_d} \end{bmatrix} \right\|_\infty < 1 \tag{3.3}$$

Remark 3.4 *There are several drawbacks to the reformulation of the parameter-dependent control design problem into the optimally scaled small-gain condition given in (3.2):*

1. *the scaled $\mathcal{H}_\infty$ norm objective does not exploit the fact that the time-varying parameters are known to be real (as opposed to complex) numbers;*
2. *in general, the optimally scaled, small-gain theorem is more conservative at bounding the induced l_2 norm under linear fractional parameter variations than using a single quadratic Lyapunov function to bound the induced l_2 norm.*

Both facts imply that our approach may result in overly conservative designs. Nonetheless, the existence of a controller K satisfying the scaled-small gain condition (3.2) can be reformulated into a finite dimensional convex feasability problem (completed in section 6), which is attractive for computation. These disadvantages are alleviated in [2], where a similar parameter-dependent control problem is considered. There, the closed-loop induced l_2 norm is bounded using a single quadratic Lyapunov function for all values of the real parameters. The disadvantage of the approach in [2] is that under general rational parameter dependence (as is considered in the current paper) the resulting synthesis optimization is convex, yet has uncountably many constraints.

4 Characterizing stability and scaled $\mathcal{H}_\infty$ norms

Consider a finite-dimensional, linear discrete-time system

$$\begin{bmatrix} x_{k+1} \\ e_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} \quad , \quad M := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbf{R}^{(n_o+n_e)\times(n_o+n_d)}$$

and the transfer function $G(z) := D + C\left(zI_n - A\right)^{-1} B$.

Theorem 4.1 *The discrete-time system has $\rho(A) < 1$ and $\|G\|_\infty < 1$ if and only if there exists an invertible matrix $J_o \in \mathbf{R}^{n_o \times n_o}$ such that*

$$\bar{\sigma}\left(\begin{bmatrix} J_o & 0 \\ 0 & I_{n_e} \end{bmatrix} M \begin{bmatrix} J_o^{-1} & 0 \\ 0 & I_{n_d} \end{bmatrix} \right) < 1.$$

Input/output scaling matrices can easily be included. In our application, the scaling matrices come from the set $\mathbf{J}(\mathcal{N}, \mathcal{R})$. For purposes that will become clear later, it will be useful to partition the state-space matrix in a slightly different manner. Consider a linear system, with three sets of inputs,

$\beta_k \in \mathbf{R}^n, \tilde{\beta}_k \in \mathbf{R}^r, d_k \in \mathbf{R}^{n_d}$, and three sets of outputs $\alpha_k \in \mathbf{R}^n, \tilde{\alpha}_k \in \mathbf{R}^r, e_k \in \mathbf{R}^{n_e}$ with state space description given by

$$\begin{bmatrix} x_{k+1} \\ \alpha_k \\ \tilde{x}_{k+1} \\ \tilde{\alpha}_k \\ e_k \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & A_{12} & B_{12} & B_{13} \\ C_{11} & D_{11} & C_{12} & D_{12} & D_{13} \\ A_{21} & B_{21} & A_{22} & B_{22} & B_{23} \\ C_{21} & D_{21} & C_{22} & D_{22} & D_{23} \\ C_{31} & D_{31} & C_{32} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} x_k \\ \beta_k \\ \tilde{x}_k \\ \tilde{\beta}_k \\ d_k \end{bmatrix} =: M \begin{bmatrix} x_k \\ \beta_k \\ \tilde{x}_k \\ \tilde{\beta}_k \\ d_k \end{bmatrix} \tag{4.1}$$

so that $M \in \mathbf{R}^{(n_o+n+r_o+r+n_e)\times(n_o+n+r_o+r+n_d)}$. Here, $x_k \in \mathbf{R}^{n_o}$ and $\tilde{x}_k \in \mathbf{R}^{r_o}$. Let G be the transfer function from $\begin{bmatrix} \beta & \tilde{\beta} & d \end{bmatrix}^T$ to $\begin{bmatrix} \alpha & \tilde{\alpha} & e \end{bmatrix}^T$. Recall that $\mathbf{J}(\mathcal{N}, \mathcal{R})$ is a set of invertible matrices, of dimension $n + r$, defined in terms of the integer f-tuples $\mathcal{N}$ and $\mathcal{R}$. Let $\overline{\mathcal{N}}$ denote the $(f + 1)$-tuple of integers $(n_o, n_1, n_2, \ldots, n_f)$, and $\overline{\mathcal{R}}$ denote the $(f + 1)$-tuple of integers $(r_o, r_1, \ldots, r_f)$. Proceeding in the same fashion, define $\mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right)$ as the set of invertible matrices with block diagonal structure given as

$$\mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right) := \left\{ \begin{bmatrix} J_{11_o} & 0 & J_{12_o} & 0 \\ 0 & J_{11} & 0 & J_{12} \\ J_{21_o} & 0 & J_{22_o} & 0 \\ 0 & J_{21} & 0 & J_{22} \end{bmatrix} : \begin{bmatrix} J_{11_o} & J_{12_o} \\ J_{21_o} & J_{22_o} \end{bmatrix} \in \mathbf{R}^{(n_o+r_o)\times(n_o+r_o)}, \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \in \mathbf{J}(\mathcal{N}, \mathcal{R}) \right\}$$

Note that $\mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right)$ has exactly the same structure as $\mathbf{J}(\mathcal{N}, \mathcal{R})$, except that it is associated with the $(f + 1)$-tuples $\overline{\mathcal{N}}$ and $\overline{\mathcal{R}}$.

Theorem 4.2 *The linear system described in (4.1) is internally exponentially stable, and there exists a $J \in \mathbf{J}(\mathcal{N}, \mathcal{R})$ such that*

$$\left\| \begin{bmatrix} J & 0 \\ 0 & I_{n_e} \end{bmatrix} G \begin{bmatrix} J^{-1} & 0 \\ 0 & I_{n_d} \end{bmatrix} \right\|_\infty < 1$$

if and only if there exists a matrix $\overline{J} \in \mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right)$ such that

$$\bar{\sigma}\left(\begin{bmatrix} \overline{J} & 0 \\ 0 & I_{n_e} \end{bmatrix} M \begin{bmatrix} \overline{J}^{-1} & 0 \\ 0 & I_{n_d} \end{bmatrix} \right) < 1.$$

Next, consider the LFT controller defined in Section 2, with $\tilde{x}_k \in \mathbf{R}^{r_o}$. Let $\mathcal{S}_K$ be the constant matrix

$$\mathcal{S}_K := \begin{bmatrix} \tilde{D}_{11} & \tilde{C}_1 & \tilde{D}_{12} \\ \tilde{B}_1 & \tilde{A} & \tilde{B}_2 \\ \tilde{D}_{21} & \tilde{C}_2 & \tilde{D}_{22} \end{bmatrix} \in \mathbf{R}^{(n_u+r_o+r)\times(n_y+r_o+r)}$$

The matrix $\mathcal{S}_K$ governs the state-evolution of the LTI portion of the controller, so that once the dimensions $r_o, r_1, \ldots, r_f$ are specified, the matrix $\mathcal{S}_K$ is a constant matrix, of known dimension, which completely represents the controller. Note that from a control design viewpoint, $\mathcal{S}_K$ may be chosen arbitrarily. Similarly, let $\mathcal{S}_{P_\mathcal{R}}$ be the constant matrix

$$\mathcal{S}_{P_\mathcal{R}} = \left[\begin{array}{ccccc|ccc} A_{ss} & A_{sp} & 0 & 0 & B_{1s} & B_{2s} & 0 & 0 \\ A_{ps} & A_{pp} & 0 & 0 & B_{1p} & B_{2p} & 0 & 0 \\ 0 & 0 & 0_{r_o} & 0 & 0 & 0 & I_{r_o} & 0 \\ 0 & 0 & 0 & 0_r & 0 & 0 & 0 & I_r \\ C_{1s} & C_{1p} & 0 & 0 & D_{11} & D_{12} & 0 & 0 \\ \hline C_{2s} & C_{2p} & 0 & 0 & D_{21} & D_{22} & 0 & 0 \\ 0 & 0 & I_{r_o} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix depends only on the known plant data $(A_{ss}, A_{sp}, \ldots, D_{22})$ and the (unknown) dimensions $r_o, r_1, \ldots r_f$. Now, a state-space description of the dynamic system shown in figure 6 can be expressed as a linear fractional transformation of the two constant matrices, $\mathcal{S}_{P_\mathcal{R}}$ and $\mathcal{S}_K$.

$$\begin{bmatrix} x_{k+1} \\ \alpha_k \\ \tilde{x}_{k+1} \\ \tilde{\alpha}_k \\ e_k \end{bmatrix} = \mathcal{F}_l\left(\mathcal{S}_{P_\mathcal{R}}, \mathcal{S}_K\right) \begin{bmatrix} x_k \\ \beta_k \\ \tilde{x}_k \\ \tilde{\beta}_k \\ d_k \end{bmatrix}$$

Hence, given the open-loop data $A_{ss}, A_{sp}, \ldots, D_{22}$ and the dimensions $n_o, n_1, \ldots n_f$, the following statements are equivalent

1. There exists an f-tuple of nonnegative integers $\mathcal{R} = (r_1, \ldots, r_f)$, a matrix $J \in \mathbf{J}(\mathcal{N}, \mathcal{R})$, and a stabilizing, finite-dimensional, linear, time-invariant controller K such that

$$\left\| \begin{bmatrix} J & 0 \\ 0 & I_{n_e} \end{bmatrix} \mathcal{F}_l\left(P_\mathcal{R}, K\right) \begin{bmatrix} J^{-1} & 0 \\ 0 & I_{n_d} \end{bmatrix} \right\|_\infty < 1$$

2. There exists an $(f+1)$-tuple of nonnegative integers $\overline{\mathcal{R}} = (r_o, r_1, \ldots, r_f)$, a matrix $Y \in \mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right)$ and a constant matrix $\mathcal{S}_K$ such that

$$\bar{\sigma}\left[\begin{bmatrix} Y^{\frac{1}{2}} & 0 \\ 0 & I_{n_e} \end{bmatrix} \mathcal{F}_l\left(\mathcal{S}_{P_\mathcal{R}}, \mathcal{S}_K\right) \begin{bmatrix} Y^{-\frac{1}{2}} & 0 \\ 0 & I_{n_d} \end{bmatrix} \right] < 1$$

The equivalence follows in two steps: first by going from a transfer function description to a state space description; second, using a permutation matrix, and the polar decomposition theorem, the scaling matrix from $\mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right)$ can be chosen Hermitian, positive definite, and the square-root parametrization may be used, [17]. Based on this reformulation, we can state a general constant matrix optimization problem, which encompasses the LFT control synthesis problem prescribed in Definition 3.3:

GIVEN: a nonnegative integer f; $(f+1)$ positive integers $n_o, n_1, \ldots, n_f$, denoted by $\overline{\mathcal{N}}$, with $\bar{n} := \sum_{j=0}^{f} n_j$; and a constant matrix $M \in \mathbf{R}^{(\bar{n}+n_e+n_y)\times(\bar{n}+n_d+n_u)}$, (with obvious partitioning)

DETERMINE if there exist: nonnegative integers $r_o, r_1, \ldots, r_f$, denoted by $\overline{\mathcal{R}}$, with $\bar{r} := \sum_{j=0}^{f} r_j$; a matrix $K \in \mathbf{R}^{(n_u+\bar{r})\times(n_y+\bar{r})}$; and a matrix $Y \in \mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right), Y = Y^T > 0$ such that

$$\bar{\sigma}\left[\begin{bmatrix} Y^{\frac{1}{2}} & 0 \\ 0 & I_{n_e} \end{bmatrix} \mathcal{F}_l\left(M_{\overline{\mathcal{R}}}, K\right) \begin{bmatrix} Y^{-\frac{1}{2}} & 0 \\ 0 & I_{n_d} \end{bmatrix} \right] < 1,$$

where

$$\mathbf{J}\left(\overline{\mathcal{N}},\overline{\mathcal{R}}\right) := \left\{ \begin{bmatrix} \mathbf{diag}\left(J_{11i}\right) & \mathbf{diag}\left(J_{12i}\right) \\ \mathbf{diag}\left(J_{21i}\right) & \mathbf{diag}\left(J_{22i}\right) \end{bmatrix} : \begin{bmatrix} J_{11i} & J_{12i} \\ J_{21i} & J_{22i} \end{bmatrix} \in \mathbf{R}^{(n_i+r_i)\times(n_i+r_i)}, i = 0,1,\ldots,f \right\}$$

and

$$M_{\overline{\mathcal{R}}} := \begin{bmatrix} M_{11} & 0 & M_{12} & M_{13} & 0 \\ 0 & 0_{\bar{r}} & 0 & 0 & I_{\bar{r}} \\ M_{21} & 0 & M_{22} & M_{23} & 0 \\ M_{31} & 0 & M_{32} & M_{33} & 0 \\ 0 & I_{\bar{r}} & 0 & 0 & 0 \end{bmatrix} \in \mathbf{R}^{(\bar{n}+\bar{r}+n_e+n_y+\bar{r})\times(\bar{n}+\bar{r}+n_d+n_u+\bar{r})}$$

The next two sections reformulate this problem into a finite dimensional convex feasability program, expressed in terms of AMIs.

5 Optimizing scaled linear fractional transformations

The primary tool for the problem considered here is that of minimizing the maximum singular value of a scaled linear fractional transformation. This problem is first reduced to an affine, rather than linear fractional, transformation, and then solved using a elementary extension to matrix dilation theory [4], [9], [19], [6]. The results we use can be proven easily using matrix theory, though operator generalizations are available. The proofs to Lemmas 5.1 and 5.2 are found in [18].

Lemma 5.1 *Let $R \in \mathbf{F}^{l\times l}, U \in \mathbf{F}^{l\times m}, T \in \mathbf{F}^{p\times m}$, and $V \in \mathbf{F}^{p\times l}$, where $m,p \leq l$. Let $\mathcal{Z} \subset \mathbf{F}^{l\times l}$ be a prescribed set of positive definite matrices. Then*

$$\inf_{\substack{K\in\mathbf{F}^{m\times p}, Z\in\mathcal{Z} \\ \det(I-TK)\neq 0}} \bar{\sigma}\left[Z^{\frac{1}{2}}\left(R+UK\left(I-TK\right)^{-1}V\right)Z^{-\frac{1}{2}}\right] = \inf_{Q\in\mathbf{F}^{m\times p}, Z\in\mathcal{Z}} \bar{\sigma}\left[Z^{\frac{1}{2}}\left(R+UQV\right)Z^{-\frac{1}{2}}\right] \tag{5.1}$$

Hence, in order to solve general linear fractional transformation optimization problems, only affine transformations need be considered. We also assume (without loss in generality) that U is full column rank, and that V is full row rank. The next lemma partially answers the synthesis question when similarity scalings are included.

Lemma 5.2 *Let R, U, V, be given as above. Suppose $U_\perp \in \mathbf{F}^{l\times(l-m)}$ and $V_\perp \in \mathbf{F}^{(l-p)\times l}$ are chosen such that $\begin{bmatrix} U & U_\perp \end{bmatrix}$, $\begin{bmatrix} V \\ V_\perp \end{bmatrix}$ are both invertible, and that $U^*U_\perp = 0_{m\times(l-m)}, VV_\perp^* = 0_{p\times(l-p)}$. Let $\mathcal{Z} \subset \mathbf{F}^{l\times l}$ be a given set of positive definite, Hermitian matrices. Then*

$$\inf_{\substack{Q\in\mathbf{F}^{m\times p} \\ Z\in\mathcal{Z}}} \bar{\sigma}\left[Z^{\frac{1}{2}}\left(R+UQV\right)Z^{-\frac{1}{2}}\right] < 1 \tag{5.2}$$

if and only if there is a $Z \in \mathcal{Z}$ such that

$$V_\perp\left(R^*ZR - Z\right)V_\perp^* < 0 \quad \text{and} \quad U_\perp^*\left(RZ^{-1}R^* - Z^{-1}\right)U_\perp < 0. \tag{5.3}$$

If these conditions are satisfied, then one suitable choice for Q is:

$$\begin{aligned}
\tilde{R} &:= Z^{\frac{1}{2}} R Z^{-\frac{1}{2}}, \quad \tilde{U} := Z^{\frac{1}{2}} U \left(U^* Z U\right)^{-\frac{1}{2}}, \quad \tilde{V} := \left(V Z^{-1} V^*\right)^{-\frac{1}{2}} V Z^{-\frac{1}{2}} \\
T_1 &:= \left(U_\perp^* Z^{-1} U_\perp\right)^{-\frac{1}{2}} U_\perp^* R V_\perp^* \left(V_\perp Z V_\perp^*\right)^{-\frac{1}{2}} \\
T_2 &:= \left(U^* Z U\right)^{-\frac{1}{2}} U^* Z R V_\perp^* \left(V_\perp Z V_\perp^*\right)^{-\frac{1}{2}} \\
T_3 &:= \left(U_\perp^* Z^{-1} U_\perp\right)^{-\frac{1}{2}} U_\perp^* R Z^{-1} V^* \left(V Z^{-1} V^*\right)^{-\frac{1}{2}} \\
Q' &:= -T_2 \left(I - T_1^* T_1\right)^{-\frac{1}{2}} T_1^* \left(I - T_1 T_1^*\right)^{-\frac{1}{2}} T_3 \\
Q &:= \left(U^* Z U\right)^{-\frac{1}{2}} \left(Q' - \tilde{U}^* \tilde{R} \tilde{V}^*\right) \left(V Z^{-1} V^*\right)^{-\frac{1}{2}}.
\end{aligned}$$

The 1st condition in (5.3) represents a convex constraint on the variable Z, and the 2nd condition in (5.3) represents a convex constraint on the variable Z^{-1}, though the two conditions together do not form a convex constraint on either variable, [16]. Hence the general problem in equation (5.2) is very difficult, and at the moment, unsolved. However, in the LFT control synthesis problem, there is sufficient additional structure to allow these two conditions to be recast as jointly convex.

6 LFT Control Synthesis Solution

In order to put the structured optimal control problem in the form considered in section 5, define $\mathcal{Z}$ in terms of $\mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right)$,

$$\mathcal{Z} := \left\{ \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} : Y \in \mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right), Y = Y^T > 0 \right\}. \tag{6.1}$$

Note that by Lemma 5.1, and the properties of $\mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right)$, the LFT synthesis problem is in the form: Is

$$\inf_{\substack{Z \in \mathcal{Z} \\ Q \in \mathcal{M}(\mathbf{R})}} \bar{\sigma}\left(Z^{\frac{1}{2}} \left(R + UQV\right) Z^{-\frac{1}{2}}\right) < 1 \text{ ?}, \tag{6.2}$$

with

$$R = \begin{bmatrix} M_{11} & 0 & M_{12} \\ 0 & 0_{\bar{r}} & 0 \\ M_{21} & 0 & M_{22} \end{bmatrix}, \quad U = \begin{bmatrix} M_{13} & 0 \\ 0 & I_{\bar{r}} \\ M_{23} & 0 \end{bmatrix}, \quad V = \begin{bmatrix} M_{31} & 0 & M_{32} \\ 0 & I_{\bar{r}} & 0 \end{bmatrix}. \tag{6.3}$$

Remark 6.1 *Note that we are sloppy about the dimension of the identity matrix in the lower corner of $\mathcal{Z}$ in (6.1). Let Z_e and Z_d represent the two "different" Z matrices (these are different only in the size of the identity block). The conditions in (5.3) are written as $U_\perp^T \left(R Z_d^{-1} R^T - Z_e^{-1}\right) U_\perp < 0$ and $V_\perp \left(R^T Z_e R - Z_d\right) V_\perp^T < 0$.*

Now, for notational purposes, define a matrix E as

$$E := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}. \tag{6.4}$$

It is clear that the matrices $U_\perp$ and $V_\perp$ must be of the form

$$U_\perp = \begin{bmatrix} U_{\perp,1} \\ 0 \\ U_{\perp,2} \end{bmatrix}, \quad V_\perp = \begin{bmatrix} V_{\perp,1} & 0 & V_{\perp,2} \end{bmatrix}. \tag{6.5}$$

Applying Lemma 5.2, there exists a matrix $Z \in \mathcal{Z}$, and a matrix $Q \in \mathbf{R}^{(n_u+\bar{r})\times(n_y+\bar{r})}$ such that

$$\bar{\sigma}\left(Z^{\frac{1}{2}}\left(R+UQV\right)Z^{-\frac{1}{2}}\right) < 1$$

if and only if there exists a matrix $Z \in \mathcal{Z}$, such that the two conditions in (5.3) hold. Denote any $Z \in \mathcal{Z}$ as $Z = \text{diag}[Y \ I]$, where $Y \in \mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right), Y = Y^T > 0$. Let X denote Y^{-1}. Note that $X \in \mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right)$ and is of the form

$$X = \begin{bmatrix} \mathbf{diag}\left(X_{11i}\right) & \mathbf{diag}\left(X_{12i}\right) \\ \mathbf{diag}\left(X_{12i}^T\right) & \mathbf{diag}\left(X_{22i}\right) \end{bmatrix}$$

where for each i

$$\begin{bmatrix} X_{11i} & X_{12i} \\ X_{12i}^T & X_{22i} \end{bmatrix} = \begin{bmatrix} Y_{11i} & Y_{12i} \\ Y_{12i}^T & Y_{22i} \end{bmatrix}^{-1}$$

Substituting the particular data for R, $U_\perp$ and $V_\perp$ into the conditions of (5.3) gives that the necessary and sufficient condition for solvability of the LFT synthesis problem is the existence of positive definite, symmetric matrices $X, Y \in \mathbf{J}\left(\overline{\mathcal{N}}, \overline{\mathcal{R}}\right)$ such that $X = Y^{-1}$, and

$$\begin{bmatrix} U_{\perp,1}^T & U_{\perp,2}^T \end{bmatrix} \left(E \begin{bmatrix} \mathbf{diag}\left(X_{11i}\right) & 0 \\ 0 & I_{n_d} \end{bmatrix} E^T - \begin{bmatrix} \mathbf{diag}\left(X_{11i}\right) & 0 \\ 0 & I_{n_e} \end{bmatrix} \right) \begin{bmatrix} U_{\perp,1} \\ U_{\perp,2} \end{bmatrix} < 0$$

$$\begin{bmatrix} V_{\perp,1} & V_{\perp,2} \end{bmatrix} \left(E^T \begin{bmatrix} \mathbf{diag}\left(Y_{11i}\right) & 0 \\ 0 & I_{n_e} \end{bmatrix} E - \begin{bmatrix} \mathbf{diag}\left(Y_{11i}\right) & 0 \\ 0 & I_{n_d} \end{bmatrix} \right) \begin{bmatrix} V_{\perp,1}^T \\ V_{\perp,2}^T \end{bmatrix} < 0$$

The constraint $X = Y^{-1}$ can be exchanged for an AMI constraint on the matrices X_{11i} and Y_{11i}.

Lemma 6.2 *Suppose that $X \in \mathbf{F}^{n\times n}$, $Y \in \mathbf{F}^{n\times n}$, with $X = X^* > 0$, and $Y = Y^* > 0$. Let r be a positive integer. Then there exists matrices $X_2 \in \mathbf{F}^{n\times r}, X_3 \in \mathbf{F}^{r\times r}$ such that $X_3 = X_3^*$, and*

$$\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} > 0 \qquad \text{and} \qquad \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & ? \\ ? & ? \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0 \qquad \text{and} \qquad \text{rank}\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \leq n + r$$

Proof: $\Leftarrow$ By assumption, there is a matrix $X_2 \in \mathbf{F}^{n\times r}$ such that $X - Y^{-1} = X_2 X_2^*$. Defining $X_3 := I_r$ completes the construction.

$\Rightarrow$ Using Schur complements, $Y = X^{-1} + X^{-1}X_2\left(X_3 - X_2^* X^{-1} X_2\right)^{-1} X_2^* X^{-1}$. Inverting, using the matrix inversion lemma, gives that $Y^{-1} = X - X_2 X_3^{-1} X_2^*$. Hence, $X - Y^{-1} = X_2 X_3^{-1} X_2^* \geq 0$, and indeed, $\text{rank}\left(X - Y^{-1}\right) = \text{rank}\left(X_2 X_3^{-1} X_2^*\right) \leq r$. ♯

Hence, given any positive definite matrices $X, Y \in \mathbf{F}^{n\times n}$, there exists an integer r such that the dilation can be completed if and only if $X - Y^{-1} \geq 0$. If this semi-definite condition holds, then the rank of $X - Y^{-1}$ determines the dimension necessary for the dilation. Since $n \times n$ matrices have rank of at most n, the maximum dimension needed for the dilation is $r_{\max} = n$, and this occurs when $X - Y^{-1} > 0$. A block-diagonal version of this lemma leads to the final result:

Theorem 6.3 *Consider the open-loop data P and $\mathcal{N}$ as in section 2. Define E, $U_\perp$ and $V_\perp$ as in equations (6.3 – 6.5). The LFT control synthesis problem is solvable if and only if there exist matrices $X_{11i}, Y_{11i} \in \mathbf{R}^{n_i \times n_i}$, $0 \le i \le f$, satisfying the convex constraints $X_{11i} = X_{11i}^T > 0, Y_{11i} = Y_{11i}^T > 0$ and*

$$\begin{bmatrix} U_{\perp,1}^T & U_{\perp,2}^T \end{bmatrix} \left(E \begin{bmatrix} \mathbf{diag}\,(X_{11i}) & 0 \\ 0 & I_{n_d} \end{bmatrix} E^T - \begin{bmatrix} \mathbf{diag}\,(X_{11i}) & 0 \\ 0 & I_{n_e} \end{bmatrix} \right) \begin{bmatrix} U_{\perp,1} \\ U_{\perp,2} \end{bmatrix} < 0$$

$$\begin{bmatrix} V_{\perp,1} & V_{\perp,2} \end{bmatrix} \left(E^T \begin{bmatrix} \mathbf{diag}\,(Y_{11i}) & 0 \\ 0 & I_{n_e} \end{bmatrix} E - \begin{bmatrix} \mathbf{diag}\,(Y_{11i}) & 0 \\ 0 & I_{n_d} \end{bmatrix} \right) \begin{bmatrix} V_{\perp,1}^T \\ V_{\perp,2}^T \end{bmatrix} < 0$$

$$\begin{bmatrix} X_{11i} & I_{n_i} \\ I_{n_i} & Y_{11i} \end{bmatrix} \ge 0 \quad \text{for } i = 0, 1, \ldots, f$$

If these convex conditions are feasable, then for each $i = 0, 1, \ldots, f$, define

$$r_i := \text{rank}\left(X_{11i} - Y_{11i}^{-1} \right) \le n_i.$$

Lemma 6.2 allows one to construct the matrices $X_{12i}, Y_{12i} \in \mathbf{R}^{n_i \times r_i}$ and matrices $X_{22i}, Y_{22i} \in \mathbf{R}^{r_i \times r_i}$ which are the missing components of the $(n_i + r_i) \times (n_i + r_i)$ block of the scaling matrix. Then, simple backsubstitution into Lemma 5.2, and Lemma 5.1 produces the state-space matrix $\mathcal{S}_K$ of the parameter-dependent controller. As noted, the state dimension of the controller is r_o, which is the rank of $X_{11o} - Y_{11o}^{-1}$. Necessarily, this rank is less than or equal to n_o, the state dimension of the plant. Similar for each δ_i parameter – the number of times that δ_i must appear in the controller, r_i, is necessarily less than or equal to the number of times it appears in the plant, n_i.

Remark 6.4 *The feasability of the convex set of inequalities in Theorem 6.3 can be tested using the special methods found in [14], [3] and [15], as well as general nondifferentiable convex optimization methods.*

Remark 6.5 *The result has been derived for* **real** *open-loop state-space data ($\mathcal{S}_{P_\mathcal{R}} \in \mathbf{R}^{(\cdot \times \cdot)}$) and real controllers ($\mathcal{S}_K \in \mathbf{R}^{(\cdot \times \cdot)}$) (and consequently, real scalings). For a complex version, simply replace $\mathbf{R}$ by $\mathbf{C}$, and transposes by complex-conjugate-transposes everywhere, and everything remains true. Also, the LFT parameter-dependent problem addressed here included only linear fractional dependence on repeated-scalar parameters. Alternatively, it is simple to also include linear fractional dependence on repeated full-block (rectangular matrices) parameters, at the expense of additional notation. Qualitatively, the results one obtains are identical to those presented here.*

Remark 6.6 *One question remains: Is the controller, composed of the linear, time-invariant K and the time-varying $\mathcal{L}_\mathcal{R}[\delta]$ itself a causal map from y to u? Note that if condition (3.2) is true, then the interconnection in Figure 4 is well-posed. That is, for any admissible δ sequence, and any sequence d, there exist unique signals $e, u, y, \alpha, \beta, \tilde{\alpha}, \tilde{\beta}$, solving the loop equations, and moreover, these signals depend causally on d and δ. However, this does not guarantee that the interconnection in Figure 2 is well-posed. If $I - \mathcal{L}_\mathcal{R}[\delta(k)]\tilde{D}_{22}$ is invertible at each time, then there are no invertibility problems, and causality follows. In fact, the controller can be directly implemented as in Figure 2. However, the controller is designed based on the reparametrization in Lemma 5.1 and the affine matrix inequality optimization of Theorem 6.3. It is impossible to constrain the details of the $\tilde{D}_{22}$ portion of the matrix $\mathcal{S}_K$ that solves the optimization. Hence, it may be that $I - \tilde{D}_{22}\mathcal{L}_\mathcal{R}[\delta(k)]$ is not invertible for some k. In general, this implies that for a given controller state $\tilde{x}_k$ and measurement y_k, there are multiple solutions to the loop equations in equation (2.1) (alternatively, there might be no solutions, but the well-posedness of Figure 4 prevents this). So, there seems to be some potential problems with* *implementing equations (2.1). However, note that if equation (3.2) is satisfied, then using the same K and J, for some $\epsilon > 0$, the norm of the system shown in Figure 7 is still less than 1. This follows by continuity, since the interconnection is internally stable. Manipulating the block diagrams in Figures 7, 4 and 2 shows that the extra input/output channel is in fact an additive perturbation around the controller's LFT dependence. The small-gain theorem applied to Figure 7 implies that small deviations can be made in the controller LFT implementation in Figure 8, while preserving the guaranteed bound on induced l_2 performance in Figure 3. These small deviations are effected by the additive perturbation $\epsilon^2 \Delta_k$, so that $\tilde{\beta}_k = \left(\mathcal{L}_{\mathcal{R}}\left[\delta(k)\right] + \epsilon^2 \Delta_k\right) \tilde{\alpha}_k$. The only constraint is $\bar{\sigma}\left(\Delta_k\right) \leq 1$, which ensures that the modified controller equations can be directly implemented in a well-posed manner.*

Remark 6.7 *Consider the standard $\mathcal{H}_\infty$ optimal control problem, and for simplicity in the formulae, assume that all of the orthogonality assumptions are in place, [7]. If one performs a bilinear transformation on the data, and applies the results of Theorem 6.3 to the transformed data (with $f = 0$) then the necessary and sufficient condition for the existence of an LTI controller which makes the closed-loop $\mathcal{H}_\infty$ norm from $d \to e$ less than 1 is the existence of positive definite matrices $X, Y \in \mathbf{R}^{n \times n}$ such that*

$$\begin{bmatrix} AX + XA^T + B_1 B_1^T - B_2 B_2^T & XC_1^T \\ C_1 X & -I \end{bmatrix} < 0 \tag{6.6}$$

$$\begin{bmatrix} A^T Y + YA + C_1^T C_1 - C_2^T C_2 & YB_1 \\ B_1^T Y & -I \end{bmatrix} < 0 \tag{6.7}$$

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0 \tag{6.8}$$

This type of result is well-known, and versions can be found in [8], [22], [21] and [2]. Expressed in terms of convex inequalities, it gives computable necessary and sufficient conditions for the existence of a FDLTI controller which achieves a closed-loop transfer function from $d \to e$ with $\mathcal{H}_\infty$ norm less than 1. There are no apriori assumptions about the open-loop plant, such as location of zeros, etc. The X and Y in equations (6.6-6.8) are not the stabilizing solutions to the Algebraic Riccati Equations (ARE) in [7], though X^{-1} and Y^{-1} are related to the ARE solutions. A lucid presentation of these relationships is found in [22]. These relationships allow one to translate between the conditions of [7], which involve positive semidefinite stabilizing solutions of AREs, and the conditions given in (6.6 - 6.8).

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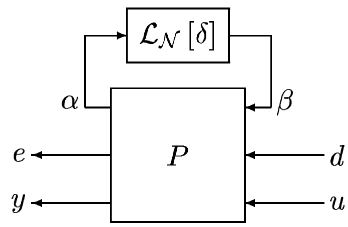

Figure 1: Parameter-dependent Plant

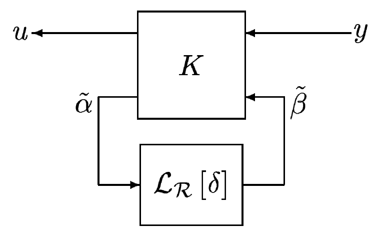

Figure 2: Parameter-dependent Controller

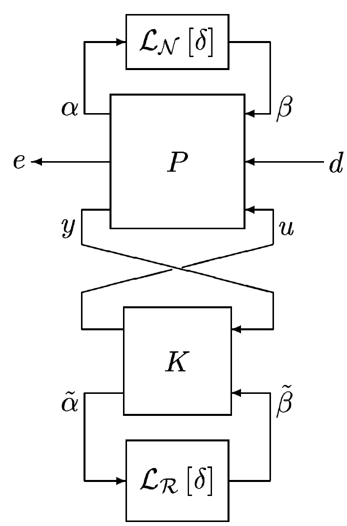

Figure 3: Closed-Loop System

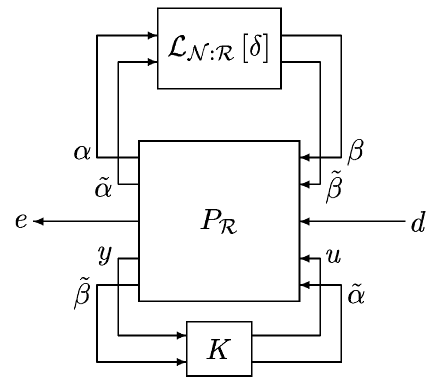

Figure 4: Parameter-dependent closed-loop system, redrawn

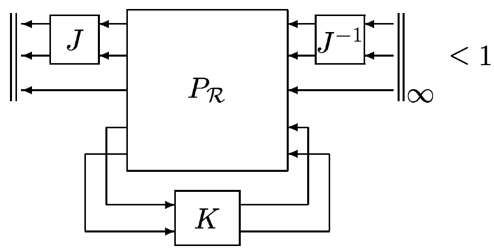

Figure 5: Scaled Small-Gain Condition

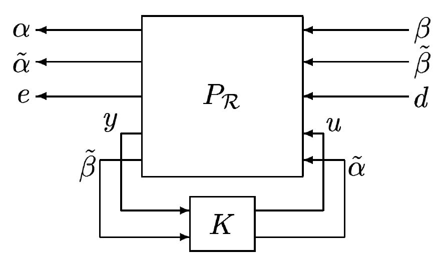

Figure 6: Closed-loop system for analysis

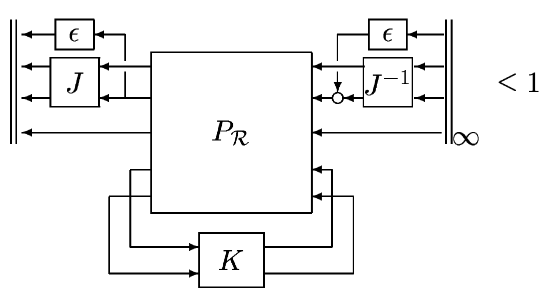

Figure 7: Perturbed scaled small-gain condition

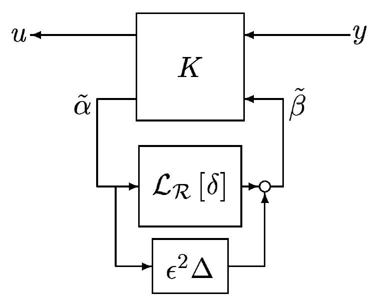

Figure 8: Perturbed parameter-dependent controller