Paper ID: ACC02-IEEE1520

Disturbance preview and on-line optimization to improve system performance

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Abstract

We present a simple, self-contained formulation of a performance enhancing, stability preserving, receding horizon control strategy for a system where preview information is available for the disturbances. The simplicity of the derivation is due to (and its benefits somewhat offset by) a set of stringent and highly structured assumptions. The formulation has two notable features: it uses a suboptimal value function for terminal cost, and relies on optimization strategies that only require a trivial improvement property, allowing implementation as an "anytime" algorithm. An example of tracking control of an air-to-air missile illustrates the possible benefits of this control methodology.

1 Introduction

The availability of faster and cheaper microprocessors has made it possible to implement automatic control algorithms which are much more complex and computationally intensive than anything that could have been implemented a generation ago. The control algorithm development presented here is a version of receding horizon control and is computationally intensive, and, in some cases, may offer significant performance increases. Recently, many groups have developed theoretical results to prove the stabilizing and performance enhancing properties of the receding horizon approach [1], [2], [3], [9] - [15], [17], [20]. By combining the ideas from [18] and [4] for continuous-time nonlinear systems, the work of Jadbabaie, Yu, and Hauser [7, 8] provides the basis for most of the results in this paper.

We extend the methods of receding horizon control to the case where a discrete nonlinear dynamic system is driven by disturbances, and where previews of these disturbances are available. Our approach uses a suboptimal value function as a terminal cost. Also, we only require improvement, not optimality, in the optimization step so that we can handle local minima or reduced computational resources. The ability to terminate the optimization early makes our approach implementable as an "anytime" algorithm.

In §2 we introduce the dynamic system and our assumptions. Sections 3, 4 give the control objective and algorithm. In §5 an important lemma unlocks the performance and stability results that are shown in §6. We formulate a missile tracking problem in §7; the results and a comparison with the legacy controller are in §8.

2 Definitions, problem setup and assumptions

First, we need a few mathematical preliminaries and notational conventions for our dynamic system $x_{k+1} = f(x_k, w_k, u_k)$, where $x_k \in \mathbb{R}^n$ with disturbance $w_k \in \mathbb{R}^l$, and control input $u_k \in \mathbb{R}^m$.

$\mathcal{K}$-functions: $\alpha(\cdot) : [0, a) \to [0, \infty)$ is a $\mathcal{K}$-function if it is continuous and strictly increasing on $[0, a)$ and $\alpha(0) = 0$.

Balls: For $r \geq 0$ and $n \in \mathbb{Z}^+$, $B_r^n := \{x \in \mathbb{R}^n : \|x\| \leq r\}$. When n is clear from context write B_r.

$w_{[k,k+N-1]}$: Shorthand for the sequence $\{w_j\}_{j=k}^{k+N-1}$.

l_{2+} **Spaces** : $\sum_{k=0}^{\infty} \|x_k\|^2 < \infty \Rightarrow x \in l_{2+}$.

The functions that define the problem and their assumptions follow

$f : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^n$: $x_{k+1} = f(x_k, w_k, u_k)$ with $f(0,0,0) = 0$ and f continuous on $\mathbb{R}^n \times 0 \times \mathbb{R}^m$.

$h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$: The error function, continuous with $h(0,0) = 0$. Also, $\exists$ $\mathcal{K}$-function $\tau(\cdot)$, $h(x,u) > h(x,0) \geq \tau(\|x\|)$, $\forall x \in \mathbb{R}^n$, $\forall u \neq 0 \in \mathbb{R}^m$. Additionally, $\sum_{k=k_0}^{\infty} h(x_k, u_k) < \infty \Leftrightarrow x \in l_{2+}$, $u \in l_{2+}$.

$g : \mathbb{R}^l \to \mathbb{R}^+$: The effect of the disturbance on the cost, a continuous function with $g(0) = 0$, and $\exists \alpha_1 > 0 \in \mathbb{R}$ such that $g(w) \geq \alpha_1 \|w\|^2$ $\forall w \in \mathbb{R}^l$. Additionally, $\forall w \in l_{2+}$, $\sum_{k=k_0}^{\infty} g(w_k) < \infty$.

$\phi_N : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^{l \times N} \times \mathbb{R}^{m \times N} \to \mathbb{R}^n$: The system flow function, which takes the system's state forward N steps in time $x_{k_0+N} = \phi_N(k_0, x_{k_0}, w_{[k_0,k_0+N-1]}, u_{[k_0,k_0+N-1]})$.

$\mu : \mathbb{R}^n \to \mathbb{R}^m$ The baseline globally stabilizing controller, referred to as the legacy controller, which meets the following norm bound. $\exists r > 0$ such that for each N, $\exists \sigma_N \in \mathcal{K}$: $\forall x_k \in B_r^n$, $\|\{\mu(\phi_j(k, x_k, 0, \{\mu(x_l)\}_{l=k}^{k+j-1}))\}_{j=0}^{N-1}\| \leq \sigma_N(\|x_k\|)$.

$V : \mathbb{R}^n \to \mathbb{R}^+$: V is a continuous positive definite function such that $\forall w \in \mathbb{R}^l$, $\forall x \in \mathbb{R}^n$

$$V\Big(f(x, w, \mu(x))\Big) + h(x, \mu(x)) - g(w) \leq V(x).$$

Recursively on the flow of the system, the above gives $V(x_{k_0}) \geq \sum_{k=k_0}^{\infty} \Big(h(x_k, \mu(x_k)) - g(w_k)\Big)$, $\forall w$, $\forall x_{k_0}$. This shows $V(\xi) \geq \mathcal{M}_\mu(\xi)$ with

$$\mathcal{M}_\mu(\xi) := \sup_w \sum_{k=k_0}^{\infty} \left(h(x_k, \mu(x_k)) - g(w_k)\right) \quad (1)$$

subject to $x_{k_0} = \xi$ and $x_{k+1} = f(x_k, w_k, \mu(x_k))$.

The bound on $\mathcal{M}_\mu(\xi)$ above implicitly shows that the value function is well defined $\forall\xi$ with the controller μ, which is a rather strong statement about the quality of the legacy controller.

$\mathcal{L}_N : \mathbb{R}^n \times \mathbb{R}^{l\times N} \times \mathbb{R}^{m\times N} \to \mathbb{R}$: The cost-like function

$$\mathcal{L}_N(x_{k_0}, w_{[k_0,k_0+N-1]}, u_{[k_0,k_0+N-1]}) := \sum_{k=k_0}^{k_0+N-1}\Big[h\Big(\phi_{k-k_0}(k_0, x_{k_0}, w_{[k_0,k-1]}, u_{[k_0,k-1]}), u_k\Big) - g(w_k)\Big] + V\Big(\phi_N(k_0, x_{k_0}, w_{[k_0,k_0+N-1]}, u_{[k_0,k_0+N-1]})\Big)$$

$\mathcal{A}_C : \mathbb{R}^{m\times N} \to 2^{\mathbb{R}^{m\times N}}$: The optimization engine, with set valued range, gives trivial improvement

$$v \in \mathcal{A}_C(u) \Rightarrow C(v) \le C(u)$$

We will use $C(\cdot) = \mathcal{L}_N(x_{k_0}, w_{[k_0,k_0+N-1]}, \cdot)$

$\mathcal{S}_\mu : \mathbb{R}^n \times \mathbb{R}^{l\times N} \times \mathbb{R}^{m\times N} \to \mathbb{R}^{m\times N}$: A control sequence time shift that appends a control action from the baseline controller.

$$\mathcal{S}_\mu(x_k, w_{[k,k+N-1]}, u_{[k,k+N-1]}) := \Big\{u_{[k+1,k+N-1]}, \mu\Big(\phi_N\Big(k, x_k, w_{[k,k+N-1]}, u_{[k,k+N-1]}\Big)\Big)\Big\}$$

3 Control Objective

Using the legacy controller, the worst case cost incurred starting from some state $x_k = \xi$ is given by $\mathcal{M}_\mu(\xi)$, and our goal is to use N steps of preview information about the disturbance, $w^p_{[k,k+N-1]}$, to choose a value of u_k that results in a lower cost and retains a guarantee of stability. We use $\mathcal{A}_C$, the optimization engine, to minimize $\mathcal{L}_N$, the incurred cost over an N step horizon with a terminal state worst case disturbance penalty.

4 The Control Strategy

With the framework above, we can define the suboptimal look-ahead disturbance rejection receding horizon control algorithm (RHC algorithm).

RHC algorithm: With an N-step preview of the disturbance, $w^p_{[k,k+N-1]}$, available

1. (a) Start-up, if $k = k_0$,
   $$\hat{u} = \{\mu(\phi_j(k, x_k, 0, \{\mu(x_l)\}_{l=k}^{k+j-1}))\}_{j=0}^{N-1}$$
   $$u^{\bar{\star},k_0} \in \mathcal{A}_{\mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]},\cdot)}(\hat{u})$$
   (b) Running
   $$\hat{u} = \mathcal{S}_\mu(x_{k-1}, w^p_{[k-1,k+N-2]}, u^{\bar{\star},k-1})$$
   $$u^{\bar{\star},k} \in \mathcal{A}_{\mathcal{L}_N(x_k, w^p_{[k,k+N-1]},\cdot)}\Big(\hat{u}\Big)$$
2. Set $u_k = u_k^{\bar{\star},k}$, increment k and repeat.

5 A Keystone Lemma

The following lemma shows that the cost-to-go of implementing one step of the control strategy and re-optimizing over the next N steps is no greater than the N step preview cost starting from x_{k_0}. This lemma acts as a keystone in all our derivations.

Lemma 1 *Under the control strategy defined above, for a given $w^p_{[k_0,k_0+N]}$ and x_{k_0},*

$$\mathcal{L}_N(x_{k_0+1}, w^p_{[k_0+1,k_0+N]}, u^{\bar{\star},k_0+1}) + h(x_{k_0}, u_{k_0}^{\bar{\star},k_0}) - g(w^p_{k_0}) \le \mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0})$$

Proof:
Let $\bar{u}^{k_0+1}_{[k_0+1,k_0+N]} = \mathcal{S}_\mu(x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}_{[k_0,k_0+N-1]})$, and note that $\bar{u}_j^{k_0+1} := u_j^{\bar{\star},k_0}$, $\forall j \in \{k_0+1, k_0+N-1\}$. Following the definition and manipulating we get the equation chain below, where for size we let $k_1 := k_0+1$.

$$\begin{aligned}
&\mathcal{L}_N(x_{k_1}, w^p_{[k_1,k_0+N]}, \bar{u}^{k_1}) \\
&:= \sum_{k=k_1}^{k_0+N}\Big[h\Big(\underbrace{\phi_{k-k_1}(k_1, x_{k_1}, w^p_{[k_1,k-1]}, \bar{u}^{k_1}_{[k_1,k-1]})}_{x_k,\ \text{if } \bar{u}^{k-1} \text{ and } w^p \text{ are used}}, \bar{u}^{k_1}_k\Big) - g(w^p_k)\Big] \\
&\quad + V\Big(\underbrace{\phi_N(k_1, x_{k_1}, w^p_{[k_1,k_0+N]}, \bar{u}^{k_1}_{[k_1,k_0+N]})}_{x_{k_0+N+1} \text{ if } \bar{u}^{k-1} \text{ and } w^p \text{ are used}}\Big) \\
&= -(h(x_{k_0}, u^{\bar{\star},k_0}_{k_0}) - g(w^p_{k_0})) + (h(x_{k_0}, u^{\bar{\star},k_0}_{k_0}) - g(w^p_{k_0})) \\
&\quad + \sum_{k=k_1}^{k_0+N-1}\Big[h\Big(\underbrace{\phi_{k-k_1}(k_1, x_{k_1}, w^p_{[k_1,k-1]}, u^{\bar{\star},k_0}_{[k_1,k-1]})}_{x_k \text{ if } u^{\bar{\star},k_0} \text{ and } w^p \text{ are used}}, u^{\bar{\star},k_0}_k\Big) - g(w^p_k)\Big] \\
&\quad + h\Big(\underbrace{\phi_{N-1}(k_1, x_{k_1}, w^p_{[k_1,k_0+N-1]}, \bar{u}^{k_1}_{[k_1,k_0+N-1]})}_{x_{k_0+N} \text{ if } \bar{u}^{k-1} \text{ and } w^p \text{ are used}}, \bar{u}^{k_1}_{k_0+N}\Big) \\
&\quad - g(w^p_{k_0+N}) + V\Big(\underbrace{\phi_N(k_1, x_{k_1}, w^p_{[k_1,k_0+N]}, \bar{u}^{k_1}_{[k_1,k_0+N]})}_{x_{k_0+N+1} \text{ if } \bar{u}^{k-1} \text{ and } w^p \text{ are used}}\Big) \\
&= -\Big(h(x_{k_0}, u^{\bar{\star},k_0}_{k_0}) - g(w^p_{k_0})\Big) \\
&\quad + \sum_{k=k_0}^{k_0+N-1}\Big[h\Big(\underbrace{\phi_{k-k_0}(k_0, x_{k_0}, w^p_{[k_0,k-1]}, u^{\bar{\star},k_0}_{[k_0,k-1]})}_{x_k \text{ if } u^{\bar{\star},k_0} \text{ and } w^p \text{ are used}}, u^{\bar{\star},k_0}_k\Big) - g(w^p_k)\Big] \\
&\quad + h\Big(\underbrace{\phi_N(k_0, x_{k_0}, w^p_{[k_0,k_0+N-1]}, \bar{u}^{\bar{\star},k_0}_{[k_0,k_0+N-1]})}_{x_{k_0+N} \text{ if } u^{\bar{\star},k_0} \text{ and } w^p \text{ are used}}, \bar{u}^{k_1}_{k_0+N}\Big) \\
&\quad - g(w^p_{k_0+N}) + V\Big(\underbrace{\phi_N(k_1, x_{k_1}, w^p_{[k_1,k_0+N]}, \bar{u}^{k_1}_{[k_1,k_0+N]})}_{x_{k_0+N+1} \text{ if } \bar{u}^{k-1} \text{ and } w^p \text{ are used}}\Big) \\
&\overset{(a)}{\le} -\Big(h(x_{k_0}, u^{\bar{\star},k_0}_{k_0}) - g(w^p_{k_0})\Big) \\
&\quad + \sum_{k=k_0}^{k_0+N-1}\Big[h\Big(\underbrace{\phi_{k-k_0}(k_0, x_{k_0}, w^p_{[k_0,k-1]}, u^{\bar{\star},k_0}_{[k_0,k-1]})}_{x_k \text{ if } u^{\bar{\star},k_0} \text{ and } w^p \text{ are used}}, u^{\bar{\star},k_0}_k\Big) - g(w^p_k)\Big] \\
&\quad + V\Big(\underbrace{\phi_N(k_0, x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}_{[k_0,k_0+N-1]})}_{x_{k_0+N} \text{ if } u^{\bar{\star},k_0} \text{ and } w^p \text{ are used}}\Big) \\
&= -\Big(h(x_{k_0}, u^{\bar{\star},k_0}_{k_0}) - g(w^p_{k_0})\Big) + \mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0})
\end{aligned}$$

(a) is by the property of V from its definition, which is valid here since the control at k_0+N is from μ. By the sub-optimality of $\bar{u}$ we know that

$$\mathcal{L}_N(x_{k_0+1}, w^p_{[k_0+1,k_0+N]}, u^{\bar{\star},k_0+1}) \le \mathcal{L}_N(x_{k_0+1}, w^p_{[k_0+1,k_0+N]}, \bar{u}^{k_0+1})$$

which, given the above chain of relations, leads to the desired result. ∎

6 Performance and stability

By bounding the cost we can show that the RHC strategy achieves a cost that is no greater than our upper bound for the cost under the legacy controller.

Theorem 1 *$\forall w^p \in l_{2+}$, the cost resulting from the application of the RHC algorithm is bounded as follows*

$$\sum_{k=k_0}^{\infty} \left(h(x_k, u^{\bar{\star},k}) - g(w^p_k)\right) \le V(x_{k_0})$$

Proof: Let $\epsilon > 0$ and $w^p \in l_{2+}$, from Lemma 1

$$\mathcal{L}_N(x_{k_0+1}, w^p_{[k_0+1,k_0+N]}, u^{\bar{\star},k_0+1}) \le -h(x_{k_0}, u^{\bar{\star},k_0}_{k_0}) - g(w^p_{k_0}) + \mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0})$$

Taking this relation for $x_{k_0}, \ldots, x_{k_0+L-2}$, $L > 0$ and summing and canceling we have, with $k_L := k_0 + L - 1$

$$\mathcal{L}_N(x_{k_L}, w^p_{[k_L,k_L+N-1]}, u^{\bar{\star},k_L}) \le -\sum_{k=k_0}^{k_0+L-2} \left[h(x_k, u^{\bar{\star},k}_k) - g(w^p_k)\right] + \mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0})$$

If we rearrange the previous equation and use the definition of $\mathcal{L}_N(x_{k_L}, w^p_{[k_L,k_L+N-1]}, u^{\bar{\star},k_L})$ we have

$$\begin{aligned}
&\mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}) \\
&\ge \sum_{k=k_0}^{k_0+L-2} \left(h(x_k, u^{\bar{\star},k}_k) - g(w^p_k)\right) + \sum_{k=k_L}^{k_L+N-1} \Big[-g(w^p_k) \\
&\quad + h(\underbrace{\phi_{k-k_L}(k_L, x_{k_L}, w^p_{[k_L,k-1]}, u^{\bar{\star},k_L}_{[k_L,k-1]})}_{x_k \text{ if } u^{\bar{\star},k_L} \text{ and } w^p \text{ are used}}, u^{\bar{\star},k_L}_k)\Big] \\
&\quad + V\Big(\underbrace{\phi_N(k_L, x_{k_L}, w^p_{[k_L,k_L+N-1]}, u^{\bar{\star},k_L}_{[k_L,k_L+N-1]})}_{x_{k_0+L-1+N} \text{ if } u^{\bar{\star},k_L} \text{ and } w^p \text{ are used}}\Big)
\end{aligned}$$

Since $V \ge 0$ and $h \ge 0$, we have the following

$$\mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}) \ge \sum_{k=k_0}^{k_0+L-2} \left(h(x_k, u^{\bar{\star},k}_k) - g(w^p_k)\right) - \sum_{k=k_0+L-1}^{k_0+L-1+N-1} g(w^p_k)$$

By our assumptions on g we know that $\sum g$ converges and $g(w) \ge 0$, $\forall w$, thus, for L large enough

$$\mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}) + \epsilon \ge \sum_{k=k_0}^{k_0+L-1} \left[h(x_k, u^{\bar{\star},k}_k) - g(w^p_k)\right]$$

Taking $L \to \infty$ and rearranging yields

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \left(h(x_k, u^{\bar{\star},k}_k) - g(w^p_k)\right) \\
&\le \mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]}, u^{\bar{\star},k_0}) + \epsilon \\
&\overset{(a)}{\le} \mathcal{L}_N(x_{k_0}, w^p_{[k_0,k_0+N-1]}, [\mu(x_{k_0}), \cdots, \mu(x_{k_0+N-1})]) + \epsilon \\
&\overset{(b)}{\le} V(x_{k_0}) + \epsilon
\end{aligned}$$

Inequality (a) comes from the use of $\mathcal{A}_{\mathcal{L}_N}$ to get $u^{\bar{\star},k_0}$ from the control sequence under μ. (b) results from V's being an upper bound on the value function. ∎

Lemma 2 *Using the RHC algorithm with any known disturbance trajectory $w^p \in l_{2+}$ and any initial condition $x_{k_0} \in \mathbb{R}^n$ results in a state trajectory in l_{2+}.*

Proof: Theorem 1 gives us

$$V(x_{k_0}) \ge \sum_{k=k_0}^{\infty} \left(h(x_k, u^{\bar{\star},k}_k) - g(w^p_k)\right)$$

We know that for bounded x, $V(x) < \infty$, and by assumption $\sum g$ converges for any $w^p \in l_{2+}$. Yielding

$$\sum_{k=k_0}^{\infty} h(x_k, u^{\bar{\star},k}_k) < \infty$$

which by the assumptions on h gives $x \in l_{2+}$. ∎

This gives the result that if $w^p \in l_{2+}$ then $x \in l_{2+}$, implying $x_k \to 0$ as $k \to \infty$, and we can use result in the following theorem to have global asymptotic stability.

Theorem 2 *The system $x_{k+1} = f(x_k, u_k)$ is globally asymptotically stable under the RHC algorithm.*

Proof: This proof is based on the time-varying Lyapunov stability proof in [19]. First, a few results are needed when $w^p := 0$.

Using $V(0) = 0$ and the continuity of $\mathcal{L}_n$ allows us to upper bound $\mathcal{L}_N$ in some ball around the origin, so there exists some $\mathcal{K}$-function β and some $r_1 > 0$ such that $\forall \xi \in B^n_{r_1}$, $\forall \eta \in B^{m\times N}_{r_1}$

$$\mathcal{L}_N(\xi, 0, \eta) \le \beta(\|\xi\| + \|\eta\|) \tag{2}$$

Using the positivity of V and h along with the other assumptions on h we get $\forall \xi \in \mathbb{R}^n$, $\forall \eta \in \mathbb{R}^{m\times N}$

$$\mathcal{L}_N(\xi, 0, \eta) \ge h(\xi, \eta_{[k]}) \ge h(\xi, 0) \ge \tau(\|\xi\|) \tag{3}$$

Recall from the assumptions on μ that $\forall x_k \in B^n_r$

$$\|\mu(x_k)_{[k,k+N-1]}\| \le \sigma(\|x_k\|) \tag{4}$$

From the keystone lemma, under RHC $\forall k \ge k_0$

$$\mathcal{L}_N(x_k, 0, u^{\bar{\star},k}_k) \le \mathcal{L}_N(x_{k_0}, 0, u^{\bar{\star},k_0}_{k_0}) \tag{5}$$

Now, for any $\epsilon > 0$ pick δ_ϵ so that

$$\delta_\epsilon < \min(r, r_1) \tag{6}$$

$$\sigma(\delta_\epsilon) < r_1 \tag{7}$$

$$\beta(\delta_\epsilon + \sigma(\delta_\epsilon)) < \tau(\epsilon) \tag{8}$$

Then for any $x_{k_0} \in B^n_{\delta_\epsilon}$ we have

$$\|\mu(x_{k_0})_{[k_0,k_0+N-1]}\| \overset{(4)}{\leq} \sigma(\|x_{k_0}\|) \leq \sigma(\delta_\epsilon) \overset{(7)}{<} r_1$$

which implies that $\mu(x_{k_0})_{[k_0,k_0+N-1]} \in B^{m\times N}_{r_1}$. We can now build the following chain for all $k \geq k_0$

$$\begin{aligned}
\tau(\|x_k\|) &\overset{(3)}{\leq} \mathcal{L}_N(x_k, 0, u_k^{\bar{\star},k}) \\
&\overset{(5)}{\leq} \mathcal{L}_N(x_{k_0}, 0, u_{k_0}^{\bar{\star},k_0}) \\
&\overset{(2)}{\leq} \mathcal{L}_N(x_{k_0}, 0, \mu(x_{k_0})_{[k_0,k_0+N-1]}) \\
&\overset{(4)}{\leq} \beta(\|x_{k_0}\| + \|\mu(x_{k_0})_{[k_0,k_0+N-1]}\|) \\
&\leq \beta(\delta_\epsilon + \sigma(\delta_\epsilon)) \\
&\overset{(8)}{<} \tau(\epsilon)
\end{aligned}$$

showing that if $x_{k_0} \in B^n_{\delta_\epsilon}$ with δ_ϵ defined as in (6)-(8) then $x_k \in B^n_\epsilon\ \forall k \geq k_0$, yielding stability. Since $w^p := 0 \in l_{2+}$, Lemma 2 then gives $x_k \to 0$, resulting in global asymptotic stability. ■

7 Missile tracking control formulation

In this section, we illustrate the RHC algorithm on a single-input, single-output missile example, using a gain-scheduled PI controller as the legacy controller and the commanded normal acceleration as a disturbance. Our goal is to use the existing controller, the preview of the desired trajectory, and online optimization to improve the missile's tracking properties. A scenario for this is to view the desired trajectory as a disturbance that will be known a short time in advance.

The dynamics of the system include missile, actuator, and control states. The pitch axis missile model's states are angle of attack (α, `rad`) and pitch rate ($\dot{\theta}$, `rad/s`), with elevator deflection (δ, `rad`) as the input

$$\frac{d}{dt}\begin{bmatrix}\alpha\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix} -\frac{\bar{q}s}{mV}\bar{C}_{n\alpha} & 1 \\ \frac{\bar{q}sd}{I}C_m(|\alpha|) & \frac{\bar{q}sd^2}{2VI}\bar{C}_{m\dot{\theta}} \end{bmatrix}\begin{bmatrix}\alpha\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix} -\frac{\bar{q}s}{mV}\bar{C}_{n\delta} \\ \frac{\bar{q}sd}{I}\bar{C}_{m\delta}\end{bmatrix}\delta.$$

The actuator's first-order lag model is

$$\begin{aligned}
\frac{d}{dt}(\delta) &= \mathrm{sat_r}\Big(\omega_n\left[\delta_{\mathrm{cmd}} - \mathrm{sat_d}(\delta)\right]\Big) \\
\mathrm{sat}_a(x) &= \begin{cases} x & \text{if } |x| \leq a \\ a\,\mathrm{sgn}(x) & \text{else}\end{cases}
\end{aligned}$$

with $\mathrm{d} = 20\frac{\pi}{180}$ `rad` and $\mathrm{r} = 260\frac{\pi}{180}$ `rad/s` limiting elevator deflection and deflection rate, respectively. The legacy controller is a gain-scheduled PI controller, based on [16] using the popular *3-loop topology*. Its input is the vertical acceleration command (g_{cmd} in g's) and its output is elevator command (δ_{cmd})

$$\begin{aligned}
\frac{d}{dt}(\xi) &= -K_a(|\alpha|)g - \dot{\theta} + K_a(|\alpha|)K_{nc}(|\alpha|)g_{\mathrm{cmd}} \\
\delta_{\mathrm{cmd}} &= K_r(|\alpha|)\left[K_i(|\alpha|)\xi - \dot{\theta}\right] \\
g &= \frac{\bar{q}s}{32.2m}(\bar{C}_{n\delta}\delta + \bar{C}_{n\alpha}\alpha)
\end{aligned}$$

where g is the vertical acceleration of the missile.

Values for the physical parameters and environmental variables are as follows: dynamic pressure, $\bar{q} = 3121.8$ `lbf/in`^{2}; reference area, $s = 0.1364$ `ft`^{2}; mass, $m = 5.35$ `slugs`; velocity, $V = 1886.2$ `ft/s`; reference length, $d = 0.4167$ `ft`; longitudinal moment of inertia, $I = 52.695$ `slug-ft`^{2}. The values for the derivatives of the normal (subscript n) and pitch moment (subscript m) aerodynamic coefficients with respect to the second subscript (α, δ, or $\dot{\theta}$) are $\bar{C}_{n\alpha} = 30.09$, $\bar{C}_{n\delta} = -4.45$, $\bar{C}_{m\dot{\theta}} = -5998.5$, and $\bar{C}_{m\delta} = 166.67$.

The autopilot gains $K_{nc}(|\alpha|)$, $K_a(|\alpha|)$, $K_i(|\alpha|)$, and $K_r(|\alpha|)$, and the aerodynamic derivative $C_m(|\alpha|)$ are even functions of the angle of attack and are represented by lookup tables. The tables are parametrized by 5 numbers: 3 α-values ($\alpha = [0\ 5/57.3\ 1]$) in radians and 2 coefficient values (which are constant for $|\alpha| > 5°$). The values of the coefficients are shown below, and an example is plotted in Fig. 1.

$\lvert\alpha\rvert$	C_m	K_{nc}	K_a	K_i	K_r
0	109.72	1.944	0.018085	18.186	0.067252
$\left[\frac{5}{57.3}, 1\right]$	0	1.261	0.0654	8.467	0.067552

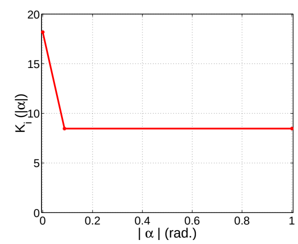

Figure 1: A plot of K_i.

Our formulation requires a static, state-feedback legacy controller, so we group the states of the PI-controller with the plant. We model the acceleration command's evolution with a first-order filter to eliminate unrealistic high frequency tracking commands.

Defining the state vector $x := [\alpha\ \dot{\theta}\ \xi\ \delta\ g_{\mathrm{cmd}}]^T$ we can form a discrete model using a first-order Euler approximation with sampling period T_s as

$$\begin{aligned}
(x_1)_{k+1} &= x_1 + T_s\left[x_2 - \frac{\bar{q}s}{mV}\left(\bar{C}_{n\alpha}x_1 + \bar{C}_{n\delta}\,\mathrm{sat_d}(x_4)\right)\right] \\
(x_2)_{k+1} &= x_2 + T_s\frac{\bar{q}sd}{I}\Big(C_m(|x_1|)x_1 + \frac{d}{2V}\bar{C}_{m\dot{\theta}}x_2 \\
&\qquad + \bar{C}_{m\delta}\,\mathrm{sat_d}(x_4)\Big) \\
(x_3)_{k+1} &= x_3 + T_s\Big[-x_2 + K_a(|x_1|)\left(K_{nc}(|x_1|)x_5 - g\right)\Big] \\
(x_4)_{k+1} &= x_4 + T_s\Big(\mathrm{sat_r}\left[\omega_n\left(u_k - \mathrm{sat_d}(x_4)\right)\right]\Big) \\
(x_5)_{k+1} &= \tau x_5 + (1-\tau)w_k
\end{aligned}$$

where the index k has been dropped from the right-hand side. The value for τ is 0.5.

Under the legacy controller, $u_k = \mu(x_k)$, where

$$\mu(x) = K_r(|x_1|)\left[K_i(|x_1|)x_3 - x_2\right].$$

We will allow the RHC algorithm to adjust the control, u_k, as given in §2. We penalize tracking error, fin deflection, and fin deflection rate with the error signal,

$$e_k = \begin{bmatrix} \frac{\bar{q}s}{32.2m}\left(\bar{C}_{n\delta}x_4 + \bar{C}_{n\alpha}x_1\right) - x_5 \\ x_4 \\ \omega_n\left(u_k - x_4\right) \end{bmatrix}.$$

Choosing error weights was not a focus of this work, but the equal weighting for tracking errors in g's, and control actions in rad and rad/s seemed reasonable, which the simulations have borne out. To match our assumptions on h and g, let $\gamma = 2$ and define

$$h(x_k, u_k) := \|e_k\|^2, \qquad g(w_k) := \gamma^2\|w_k\|^2.$$

Due to our trivial improvement property in the optimization step, we can use any canned minimization routine to act as $\mathcal{A}_C$.

The discrete time model was evaluated using a sampling time of $T_s = 0.001$ s, and all simulations were run with this timestep. The control signal of the legacy controller, whose dynamics were folded into the plant, was also updated at this rate. For the example, we took a brute-force, non-scalable approach to computing a suitable V by attempting to compute $\mathcal{M}_\mu$ on a "dense" rectangular grid. Using the iteration proposed in [6], and a finite horizon of 1200 steps, we computed approximations to the quantity in (1). Since the horizon is finite, with no guarantee that the maximum has been obtained, we have actually computed a *lower* bound for $\mathcal{M}_\mu$, and hence not a valid V. Nevertheless, we use it, knowing that the conditions (which are sufficient, not necessary) of the Theorem 1 and 2 are not satisfied. Using a PC (Pentium III, running 1.0 GHz), we computed the quantity on a 13×13×13×13×13 uniform grid with $|x_i| \leq \beta_i$, $\beta =$[.3 3 .3 .3 30], in about 3 hours. At double precision (not necessary) it takes about 3MB to store this table. This approach leaves a lot to be desired, and many alternatives could be pursued to reduce these demands. In the short term, it allowed us to complete the example and put off research on learning theory and function representation. The choice of $\gamma = 2$ was found to be sufficiently large to result in a finite-valued value function, though, for instance, 1.4 was not.

8 Missile tracking simulations and results

Results comparing the closed-loop responses of the receding horizon and the legacy controllers to square and sine wave disturbances are shown here. The receding horizon controller input was updated every 0.01 s, or every tenth step. The value function was linearly interpolated to calculate the terminal cost. The preview time was 0.1 s.

On-line optimization was done using NPSOL, [5], with numerically calculated gradients. The maximum iteration limit was set to 1; higher limits slowed the optimization considerably without yielding discernibly better performance. Each on-line optimization calculates a series of optimal inputs, $\{v_k\}_{k=k_0}^{k_0+N-1}$, for the upcoming N-step horizon. Since only the first of these, $u_{k=k_0}$, is implemented (for 0.01 s), the number of parameters in each optimization may be reduced by reducing the number of bases corresponding to the remaining 0.09 s of preview. If the preview time is 0.1 s, equivalent to 100 timesteps, then using equal bases requires optimizing over 10 parameters because the RHC input, u_k, is updated every tenth step. Instead of using 10 bases, each corresponding to 10 timesteps, to optimize the input over the 0.1 s horizon, the optimization could be done, for example, using 6 parameters (2 10-step bases followed by 4 20-step bases) or 5 parameters ([10 20 20 20 30]) and so on. The first basis should be 10 since u_k is updated every ten timesteps. The use of such unequal bases precludes the use the shift operator, $\mathcal{S}_\mu$, to provide a 'warm start' for the next optimization. For the simulations shown here, 6 bases ([10 10 20 20 20 20]) were used. The initial guess used for each optimization was the solution to the previous timestep's finite horizon optimal control problem.

All simulations were run on the nominal system as well as for 20 additional cases with uniformly distributed, random uncertainties of up to 10% in the aerodynamic coefficient derivatives ($C_m(|\alpha|)$, $\bar{C}_{n\alpha}$, $\bar{C}_{n\delta}$, $\bar{C}_{m\dot{\theta}}$, and $\bar{C}_{m\delta}$) physical parameters (s, m, d, and I) and environmental variables ($\bar{q}$ and V).

Simulations were done using a 1 Hz 30 g magnitude square wave disturbance (beginning at $t = 0.15$ s). Plots comparing the performance of the receding horizon and baseline controllers are shown in Fig. 2. Examination of the figures shows that the 0.1 s preview

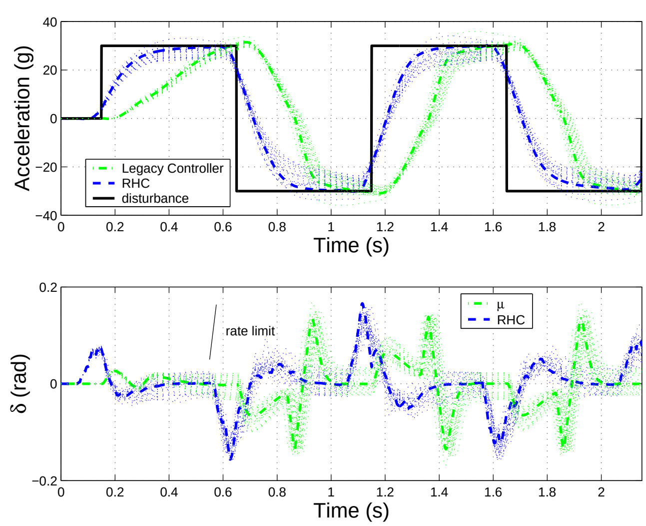

Figure 2: 30 g square wave input.

leads to anticipative action, as expected, and that the on-line optimization significantly improves the transient response as well. It is also interesting to note that the maximum fin deflection using the RHC is not much different than when using the legacy controller. The fin deflection limit is 0.349 rad, and the rate limit is shown on the plot. It is comforting to note that the RHC algorithm does not entirely break down when the parameter uncertainty is added and continues to function in a similar way.

The simulation results from applying a 1 Hz 30 g magnitude sine wave input are shown in Fig. 3. Again, as can be expected, the RHC does a better job of tracking the input than the legacy controller. In this case, however, the RHC also achieves its superior performance with a smaller maximum fin deflection by intelligently applying control effort. Again, the perturbed systems respond in a manner similar to the nominal one.

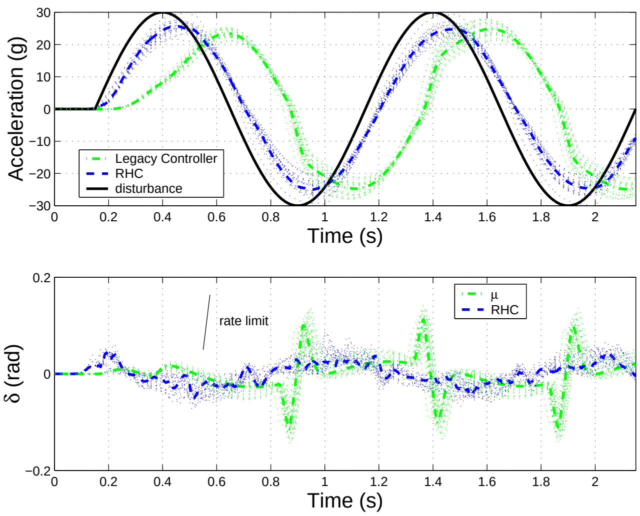

Figure 3: 30 g sine wave input.

9 Conclusions

We have shown that, in a discrete-time context, a receding horizon control algorithm can be used to take advantage of previews of exogenous signals (disturbances or tracking commands) to exhibit better performance than some nominal controller, while still guaranteeing stability of the closed-loop system.

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This work is supported by DARPA under Software Enabled Control program, USAF contract F33615-99-C-1497. The DARPA SEC program manager is Dr. John Bay, with Mr. William Koenig and Mr. Raymond Bortner (AFRL) providing technical support. Mr. Dale Van Cleave (AFRL) is the technical monitor for this contract.