Searching for Control Lyapunov Functions using Sums of Squares Programming

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Abstract

Construction of a Control Lyapunov Function (CLF) for a nonlinear system is generally a difficult problem, but once a CLF is found, stabilization of the system is straight-forward. In this paper, we present an algorithm that searches for CLFs for polynomial systems that are affine in control using sums of squares programming. We also present an algorithm for searching local CLFs for the same class of nonlinear system when global asymptotic stabilization is not possible.

1 Introduction

The work of [1] and [2] showed that for a nonlinear system that is affine in control, the existence of a smooth Control Lyapunov Function (CLF) for the system implies smooth stabilizability for the system. Given a CLF for a nonlinear system, there are several feedback laws that can stabilize the nonlinear system, one of which is given by [2]. Hence, once we have a CLF for the system, stabilization is straight-forward.

On the other hand, construction of the CLF is difficult in general, with the exception of special classes of systems. For example, [3] has shown that for a system that is feedback linearizable, a quadratic CLF can constructed in the feedback linearized coordinates. In this paper, we shall present an algorithm that that searches for a CLF for nonlinear systems that are affine in control and have polynomial vector fields using the Positivstellensatz and sums of squares(SOS) programming. We shall also present an algorithm that searches for a local CLF for nonlinear systems that cannot be globally asymptotically stabilized. Such a local CLF is also optimized to enlarge the system's region of attraction using the feedback law in [2].

The algorithms presented here are similar in flavor to our previous work [5] in that we break the optimization problem that is nonlinear in the decision polynomials into sub-problems linear in the decision polynomials by holding some of the decision polynomials fixed and optimizing over the rest of them. In [5], this involves 3 sub-steps of searching over the controller, the Lyapunov function and the SOS polynomials. In this paper, as we are not explicitly searching for the control law, our algorithm involves only a two-way search between the Lyapunov function and the SOS polynomials. Also, unlike [5], our control law here might not be a polynomial and is allowed to be non-smooth at the origin, which can be a desirable characteristic [2].

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2 Background

Suppose we are given a system of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, f and g are smooth vector fields and $f(0) = 0$.
A function V is a Control Lyapunov Function (CLF) for this system if $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth, radially unbounded, and positive definite function such that

$$\inf_{u \in \mathbb{R}} \left\{ \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u \right\} \; < \; 0 \qquad \forall\, x \neq 0 \tag{2}$$

Existence of such a V implies that (1) is globally asymptotically stabilizable at the origin.

If such a CLF is given, [2] showed how a feedback law $u = k(x)$, with $k(0) = 0$ can be constructed from the CLF such that the closed loop system is globally asymptotically stable. Under certain conditions, $k(x)$ is at least continuous at the origin and smooth everywhere else. Hence, problem of globally asymptotically stabilizing (1) is reduced to finding a CLF for the system, which is a non-trivial problem.

In subsequent sections, we shall show how we can search for a CLF, but now we shall give a quick review of the Positivstellensatz (P-satz) which will be used in our formulation.

Definition 1 *A* ***Monomial*** m_α *in* n *variables is a function defined as* $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ *for* $\alpha \in \mathbb{Z}_+^n$*. The degree of a monomial is defined,* $\deg m_\alpha := \sum_{i=1}^n \alpha_i$.

Definition 2 *A* ***Polynomial*** f *in* n *variables is a finite linear combination of monomials,*

$$f := \sum_\alpha c_\alpha m_\alpha = \sum_\alpha c_\alpha x^\alpha$$

with $c_\alpha \in \mathbb{R}$*. Define* $\mathcal{R}_n$ *to be the set of all polynomials in* n *variables. The degree of* f *is defined as* $\deg f := \max_\alpha \deg m_\alpha$ *(provided the associated* c_α *is non-zero).*
Define Σ_n to be the set of sum of squares (SOS) polynomials in n variables.

$$\Sigma_n := \left\{ p \in \mathcal{R}_n \;\middle|\; p = \sum_{i=1}^{t} f_i^2 \;,\; f_i \in \mathcal{R}_n,\, i = 1, \ldots, t \right\}$$

Obviously if $p \in \Sigma_n$, then $p(x) \geq 0 \; \forall x \in \mathbb{R}^n$.

Definition 3 *Given* $\{g_1, \ldots, g_t\} \in \mathcal{R}_n$*, the* ***Multiplicative Monoid*** *generated by* g_j*'s is the set of all finite products of* g_j*'s, including 1 (i.e. the empty product). It is denoted as* $\mathcal{M}(g_1, \ldots, g_t)$.

Definition 4 *Given* $\{f_1, \ldots, f_r\} \in \mathcal{R}_n$*, the* ***Cone*** *generated by* f_i*'s is*

$$\mathcal{P}(f_1, \ldots, f_r) := \left\{ s_0 + \sum_{i=1}^{l} s_i b_i \;\middle|\; l \in \mathbb{Z}_+, s_i \in \Sigma_n, b_i \in \mathcal{M}(f_1, \ldots, f_r) \right\}.$$

Definition 5 *Given* $\{h_1, \ldots, h_u\} \in \mathcal{R}_n$*, the* ***Ideal*** *generated by* h_k*'s is*

$$\mathcal{I}(h_1, \ldots, h_u) := \left\{ \sum h_k p_k \,\middle|\, p_k \in \mathcal{R}_n \right\}.$$

With these definitions, the Positivstellensatz theorem [6, Theorem 4.2.2] is stated below:

Theorem 1 (Positivstellensatz) *Given polynomials* $\{f_1, \ldots, f_r\}$*,* $\{g_1, \ldots, g_t\}$*, and* $\{h_1, \ldots, h_u\}$ *in* $\mathcal{R}_n$*, the following are equivalent:*

1. *The set below is empty:*

$$\{x \in \mathbb{R}^n \,|\, f_1(x) \geq 0, \ldots, f_r(x) \geq 0, g_1(x) \neq 0, \ldots, g_t(x) \neq 0, h_1(x) = 0, \ldots, h_u(x) = 0\}$$

2. *There exist polynomials* $f \in \mathcal{P}(f_1, \ldots, f_r)$*,* $g \in \mathcal{M}(g_1, \ldots, g_t)$*,* $h \in \mathcal{I}(h_1, \ldots, h_u)$ *such that*

$$f + g^2 + h = 0.$$

3 Control Lyapunov Function

In this section, we shall show how the definition of a CLF can be formulated as empty set questions so that the P-satz can be applied. This is followed by simplifications to the equations so that SOS programming can be used. Finally, we shall present the algorithm that enables us to systematically search for CLFs.

3.1 SOS formulation

A CLF must satisfy inequality (2), of which the LHS can be further evaluated as:

$$\inf_{u \in \mathbb{R}} \left\{ \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u \right\} = \begin{cases} -\infty & \text{when } \frac{\partial V}{\partial x} g(x) \neq 0 \\ \frac{\partial V}{\partial x} f(x) & \text{when } \frac{\partial V}{\partial x} g(x) = 0 \end{cases} \tag{3}$$

For a fixed x such that $\frac{\partial V}{\partial x} g(x) \neq 0$, we can make the inequality (2) hold by choosing a large value of u of the correct sign. As a result, the crucial place to check is the set of x such that $\frac{\partial V}{\partial x} g(x) = 0$. There, the inequality $\frac{\partial V}{\partial x} f(x) < 0$ must be satisfied, i.e. we want

$$\frac{\partial V}{\partial x} f(x) < 0 \;\; \forall x \in \mathbb{R}^n \;\text{ such that }\; \frac{\partial V}{\partial x} g(x) = 0,\, x \neq 0 \tag{4}$$

If we restrict (1) to f and g being polynomial vector fields, we can use SOS programming to search for a polynomial CLF V. The condition that V is positive definite and radially unbounded is rewritten as (5). Condition (4) is rewritten as an empty set condition (6) and the search for V is posed as the following feasibility problem:

$$\text{find } V \in \mathcal{R}_n \qquad \text{such that}$$

$$V(x) > 0 \;\; \forall x \in \mathbb{R}^n \backslash \{0\}, \quad V(0) = 0, \quad \text{and} \quad ||V(x)|| \to \infty \;\text{ as }\; ||x|| \to \infty \tag{5}$$

$$\left\{ x \in \mathbb{R}^n \,\middle|\, \frac{\partial V}{\partial x} g(x) = 0,\, \frac{\partial V}{\partial x} f(x) \geq 0,\, x \neq 0 \right\} \text{ is empty} \tag{6}$$

Let l_1, l_2 be positive definite, SOS polynomials that will be used to replace the non-polynomial constraints $x \neq 0$ in (5) and (6). Condition (5) can be reformulated by underbounding V by l_1, and restricting V to be a polynomial with no constant term, i.e. $V(0) = 0$. For optimization purposes, we cast the feasibility problem into a minimization problem so that the 2-way iteration (to be shown below) makes sense. We do so by introducing a scalar variable γ to minimize over. Our problem is now

$$\min \ \gamma \in \mathbb{R} \qquad \text{over } V \in \mathcal{R}_n \qquad \text{such that}$$

$$\{x \in \mathbb{R}^n \mid V(x) \leq 0,\, l_1(x) \neq 0\} \text{ is empty} \tag{7}$$

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \frac{\partial V}{\partial x} g(x) = 0,\, \frac{\partial V}{\partial x} f(x) \geq \gamma,\, l_2(x) \neq 0 \right\} \text{ is empty} \tag{8}$$

Note that V is a CLF only when we can find a non-positive γ such that (7) and (8) are satisfied. Using the P-satz, the above minimization is rewritten as follows:

$$\min \ \gamma \qquad \text{over } s_0, s_1, s_3, s_4 \in \Sigma_n, \quad V, p_2 \in \mathcal{R}_n, \quad k_1, k_2 \in \mathbb{Z}_+ \quad \text{such that}$$

$$s_3 - V s_4 + l_1^{2k_1} = 0 \tag{9}$$

$$s_0 + s_1 \left[\frac{\partial V}{\partial x} f(x) - \gamma \right] + p_2 \left[\frac{\partial V}{\partial x} g(x) \right] + l_2^{2k_2} = 0 \tag{10}$$

In order to use SOS programming tools [4], some simplifications are needed. By choosing $k_1 = k_2 = 1$, and factoring out a l_1 term in (9), and a l_2 term in (10), we get the following sufficient conditions:

$$\min \ \gamma \qquad \text{over } s_1 \in \Sigma_n,\ V, p_2 \in \mathcal{R}_n \text{ such that}$$

$$V - l_1 \in \Sigma_n \tag{11}$$

$$-\left\{ s_1 \left[\frac{\partial V}{\partial x} f(x) - \gamma \right] + p_2 \left[\frac{\partial V}{\partial x} g(x) \right] + l_2 \right\} \in \Sigma_n \tag{12}$$

Note that the constraint (12) is bilinear in the decision polynomials, which SOS programming cannot handle directly. We overcome this problem by using an iterative algorithm that alternates between searching over and holding fixed V and (s_1, p_2). Hence, at each sub-step, the problem is a linear combination of the decision polynomials, and γ will be at least monotonically non-increasing.
To ensure that (11) and (12) have a chance of being satisfied, the polynomials we are searching over must satisfy the following degree conditions:

$$\deg V = \deg l_1\,, \quad \deg(p_2 V g) \geq \deg(s_1 V f)\,, \quad \deg(p_2 V g) - 1 = \deg l_2 \tag{13}$$

Iterative CLF Search
Setup: Specify the maximum degree that will be considered for V, s_1 and p_2, while observing the constraints in (13). Set $l_1 = \epsilon \sum x_i^m$ for some small $\epsilon > 0$, and where m is the maximum degree of V. Likewise, set $l_2 = \epsilon \sum x_i^q$ for some small $\epsilon > 0$, and where q is deg($p_2 V g$) - 1. Each step of the iteration, which is indexed by i, consists of two substeps. To begin the iteration, initialize $V^{(i=0)} = \frac{1}{\epsilon} l_1$ and set the iteration index $i = 1$, and proceed to step 1.

1. Minimize γ over s_1 and p_2, with $V = V^{(i-1)}$ to obtain $s_1^{(i)}$ and $p_2^{(i)}$. A line search over γ is needed here because there is a $s_1 \gamma$ term in (12).

2. Minimize γ over V, with $s_1 = s_1^{(i)}$ and $p_2 = p_2^{(i)}$ to obtain $V^{(i)}$ and $\gamma^{(i)}$.

3. If $\gamma^{(i)} - \gamma^{(i-1)}$ is less than a specified tolerance or if $\gamma^{(i)} \leq 0$, conclude the iteration, otherwise increment i and return to substep 1.

If we can find a non-positive γ, the resulting V is a CLF. This can be seen by applying the control law proposed by [2]:

$$a(x) := \frac{\partial V}{\partial x} f(x)\ ,\ b(x) := \frac{\partial V}{\partial x} g(x)\ ,\ k(x) := \begin{cases} -\dfrac{a + \sqrt{a^2 + b^4}}{b} & \text{when} \quad b \neq 0 \\ 0 & \text{when} \quad b = 0 \end{cases} \tag{14}$$

$$\frac{dV}{dt} = a(x) + b(x)k(x) = \begin{cases} -\sqrt{a^2 + b^4} < 0 & \text{when} \quad b \neq 0 \\ a < \gamma & \text{when} \quad b = 0 \end{cases} \tag{15}$$

3.2 Multi-input Case

The formulation and algorithm presented above can be easily extended to systems with multiple inputs. Consider the following multi-input nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \tag{16}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, f and $g_1, \ldots, g_m$ are smooth vector fields and $f(0) = 0$.

For a smooth, radially unbounded, positive definite function, V, to be a CLF, it must satisfy

$$\frac{\partial V}{\partial x} f(x) < 0 \ \ \forall x \in \mathbb{R}^n \setminus \{0\} \ \text{ such that } \ \frac{\partial V}{\partial x} g_i(x) = 0,\ i = 1, \ldots, m \tag{17}$$

Equivalently, the empty set question becomes:

$$\text{Is } \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{\partial V}{\partial x} g_1(x) = 0,\ \ldots\ ,\ \frac{\partial V}{\partial x} g_m(x) = 0,\ \frac{\partial V}{\partial x} f(x) \geq 0,\ x \neq 0 \right\} \text{ empty?} \tag{18}$$

Applying P-satz to (18) and using the same simplifications as in (10), we get

$$-\left\{ s_1 \left[\frac{\partial V}{\partial x} f(x) - \gamma \right] + \sum_{i=1}^{m} p_{2i} \left[\frac{\partial V}{\partial x} g_i(x) \right] + l_2 \right\} \ \in\ \Sigma_n \tag{19}$$

For multi-input systems, we just need to replace (12) with (19) in the optimization. The number of SOS constraints still remains the same, but we need to search over polynomials $p_{21}, \ldots, p_{2m}$, instead of just p_2.

3.3 Example

The following 2nd order bilinear system is taken from [7], which has shown that this system can be globally asymptotically stabilized by an appropriate mixing of the stabilizing controllers for the slow and fast subsystems. We shall use our algorithm to find a stabilizing controller for this system without exploiting such knowledge. Also, for this example, the veracity of the solution from our algorithm can be easily checked with some simple analysis using Linear Matrix Inequalities (LMIs).

$$\begin{aligned} \dot{x}_1 &= (3x_1 + 4x_2)u \\ \dot{x}_2 &= (-20x_1 + 10x_2)u \end{aligned} \tag{20}$$

Since this system has $f = 0$, the only way that (2) can be satisfied is by finding a V such that the set $\{x \in \mathbb{R}^2 \,|\, \frac{\partial V}{\partial x} g(x) = 0\}$ is empty. We can take this analysis further by considering quadratic V. Let $V = \frac{1}{2} x^T P x$, where

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \succ 0 \tag{21}$$

Suppose we choose $\frac{\partial V}{\partial x} g(x) > 0$ for all $x \neq 0$. This condition can be reduced to an LMI:

$$\frac{\partial V}{\partial x} g(x) = (3P_{11} - 20P_{12})x_1^2 + (4P_{11} - 20P_{22} + 13P_{12})x_1 x_2 + (10P_{22} + 4P_{12})x_2^2 =: x^T M x > 0$$

$$\Leftrightarrow \qquad M := \begin{bmatrix} 3P_{11} - 20P_{12} & 2P_{11} - 10P_{22} + 6.5P_{12} \\ 2P_{11} - 10P_{22} + 6.5P_{12} & 10P_{22} + 4P_{12} \end{bmatrix} \succ 0 \tag{22}$$

We will verify that the CLFs for this system obtained from our algorithm satisfy LMIs (21) and (22).

We start our algorithm with $V_0 = \frac{1}{2}(x_1^2 + x_2^2)$, which is not a CLF because it does not satisfy (22). We set the maximum degree of V, s_1 and p_2 to be 2, 4 and 2 respectively. Our algorithm found a CLF after 2 iterations, and γ is unbounded from below. The resulting CLF is $V = 0.91719x_1^2 + 0.018365x_1x_2 + 0.31138x_2^2$, which can be easily verified that it satisfies (21) and (22).

4 Local Control Lyapunov Function

When system (1) cannot be globally asymptotically stabilized, we ask whether can we locally stabilize the system and how large can we make its region of attraction. In this section, we shall present the formulation and algorithm for searching a local CLF.

4.1 SOS Formulation

When V, a candidate CLF, fails to satisfy (2), it is because there are points x such that $\frac{\partial V}{\partial x} g(x) = 0$ and $\frac{\partial V}{\partial x} f(x) \geq 0$. If a system cannot be globally stabilized, the best that we can do is to find a compact set that excludes such points.

We want to find a level set $\Omega_\alpha = \{x \in \mathbb{R}^n \,|\, V(x) \leq \alpha\}$ such that $\forall x \in \Omega_\alpha \setminus \{0\}$ and $\frac{\partial V}{\partial x} g(x) = 0$, we have $\frac{\partial V}{\partial x} f(x) < 0$. The level set Ω_α is a region of attraction for the closed loop system when we use Sontag's feedback law (14). This is because for all $x \in \Omega_\alpha \setminus \{0\}$ such that $b = \frac{\partial V}{\partial x} g(x) \neq 0$, we have $\frac{dV}{dt} = -\sqrt{a^2 + b^4} < 0$ and when $b = 0$, $\frac{dV}{dt} = a < 0$.

To enlarge Ω_α, pick a positive definite p. Define a variable sized region $P_\beta := \{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\}$ such that $P_\beta \subseteq \Omega_\alpha$. By maximizing β, we are enlarging P_β & Ω_α.
These two conditions result in the following set containment constraints:

$$\left\{ x \in \mathbb{R}^n \middle| V(x) \leq \alpha,\, \frac{\partial V}{\partial x} g(x) = 0 \right\} \setminus \{0\} \subseteq \left\{ x \in \mathbb{R}^n \,\middle|\, \frac{\partial V}{\partial x} g(x) = 0,\, \frac{\partial V}{\partial x} f(x) < 0 \right\} \tag{23}$$

$$\{x \in \mathbb{R}^n | p(x) \le \beta\} \subseteq \{x \in \mathbb{R}^n | V(x) \le \alpha\} \tag{24}$$

The equivalent empty set questions of (23) and (24) become: Are

$$\left\{ x \in \mathbb{R}^n \,\middle|\, V(x) \le \alpha,\ \frac{\partial V}{\partial x} g(x) = 0,\ \frac{\partial V}{\partial x} f(x) \ge 0,\ x \ne 0 \right\} \quad \text{and} \tag{25}$$

$$\{x \in \mathbb{R}^n | p(x) \le \beta, V(x) \ge \alpha, V(x) \ne \alpha\} \tag{26}$$

empty? Again, using a positive definite and SOS polynomial $l_2(x)$ to replace the non-polynomial constraint $x \ne 0$ in (25) and applying the P-satz to (25) and (26), we have

$$s_0 + s_1(\alpha - V) + s_2 \left[\frac{\partial V}{\partial x} f(x)\right] + s_3(\alpha - V)\left[\frac{\partial V}{\partial x} f(x)\right] + p_4 \left[\frac{\partial V}{\partial x} g(x)\right] + l_2^{2k_2} = 0 \tag{27}$$

$$s_5 + s_6(\beta - p) + s_7(V - \alpha) + s_8(\beta - p)(V - \alpha) + (V - \alpha)^{2k_3} = 0 \tag{28}$$

By choosing $s_3 = 0$ and $k_2 = 1$, and factoring out a l_2 term, we simplify the constraint (27) into

$$-\left\{ s_1(\alpha - V) + s_2 \frac{\partial V}{\partial x} f(x) + p_4 \frac{\partial V}{\partial x} g(x) + l_2 \right\} \in \Sigma_n \tag{29}$$

Equation (28) has a $(V-\alpha)^{2k_2}$ term which cannot be optimized using SOS programming, so we cast this constraint as an $\mathcal{S}$-procedure by setting $s_5 = s_6 = 0$, $k_3 = 1$, and factoring out a $(V - \alpha)$ term.

The optimization problem for local CLF is as follows:

$$\max\ \beta \qquad \text{over } s_1, s_2, s_8 \in \Sigma_n, \quad V, p_4 \in \mathcal{R}_n \qquad \text{such that}$$

$$V - l_1 \in \Sigma_n \tag{30}$$

$$-\Big((\beta - p)s_8 + (V - \alpha)\Big) \in \Sigma_n \tag{31}$$

$$-\left\{ s_1(\alpha - V) + s_2 \frac{\partial V}{\partial x} f(x) + p_4 \frac{\partial V}{\partial x} g(x) + l_2 \right\} \in \Sigma_n \tag{32}$$

The constraint (32) is bilinear in the decision polynomials. Again, we will use an iterative algorithm that alternates between searching over and holding fixed V and (s_1, s_2, p_4). Hence, at each sub-step, the problem is a linear combination of the decision polynomials, and β will be at least monotonically non-decreasing. However, we do not have a formal stopping criteria, but a heuristics one: when the improvement of β between each iteration is less than a specified tolerance, stop. To ensure that the optimization have a chance of being feasible, the constraints must satisfy the following degree conditions:

$$\begin{gathered} \deg V = \deg l_1 \\ \deg(p s_8) \ge \deg V, \\ \left\{ \begin{array}{rcl} \deg(p_4 V g) & \ge & \deg(s_2 V f) \\ \deg(p_4 V g) - 1 & = & \deg l_2 \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{rcl} \deg(s_1 V) & \ge & \deg(s_2 V f) - 1 \\ \deg(s_1 V) & = & \deg l_2 \end{array} \right\} \end{gathered} \tag{33}$$

Iterative local CLF Search

Setup: Specify the maximum degree that will be considered for V, s_1, s_2, s_8 and p_4 while observing the constraints in (33). Set $l_1 = \epsilon \sum x_i^m$ for some small $\epsilon > 0$, and where m is the maximum degree of V. Likewise, set $l_2 = \epsilon \sum x_i^q$ for some small $\epsilon > 0$, and where q is the $\max(\deg(p_4 V g) - 1, \deg(s_1 V))$. Each step of the iteration, which is indexed by i, consists of three substeps. To begin the iteration, initialize $V^{(i=0)}$ to be any CLF for the linearized system and set the iteration index $i = 1$, and proceed to step 1.

1. Maximize α over s_1, s_2 and p_4, with $V = V^{(i-1)}$, subject to (32) to obtain $s_1^{(i)}$, $s_2^{(i)}$ and $p_4^{(i)}$. Set $(l_2^{(i)})^2 = (l_2^{(i-1)})^2/\alpha$ and $V^{(i-1)} = V^{(i-1)}/\alpha$.
2. Maximize β over s_8 with $V = V^{(i-1)}$, subject to (31) to obtain $s_8^{(i)}$.
3. Maximize β over V, with $s_1 = s_1^{(i)}$, $s_2 = s_2^{(i)}$, $s_8 = s_8^{(i)}$ and $p_4 = p_4^{(i)}$, subject to (30) - (32) to obtain $V^{(i)}$ and $\beta^{(i)}$.
4. If $\beta^{(i)} - \beta^{(i-1)}$ is less than a specified tolerance, conclude the iteration, otherwise increment i and return to substep 1.

Again, we can easily extend our formulation for local CLFs to the multi-input case as shown in Section 3.2.

4.2 Example

This example is the taken from [5]. Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_1 + \tfrac{1}{6}x_1^3 - u \end{aligned} \tag{34}$$

with $x(t) \in \mathbb{R}^2$ and $u(t) \in \mathbb{R}$.

We shall analytically show that quadratic Vs do not meet the CLF condition (2) for this system. Define $V := \frac{1}{2}x^T P x$, where P is a positive definite symmetric matrix:

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

We look at the sign of $\frac{\partial V}{\partial x} f(x)$ when $\frac{\partial V}{\partial x} g(x) = 0$:

$$\frac{\partial V}{\partial x} f(x) = x^T P f = (P_{12}x_1 + P_{22}x_2)(-x_1 + \tfrac{1}{6}x_1^3) \tag{35}$$

$$\frac{\partial V}{\partial x} g(x) = x^T P g = (P_{11} - P_{12})x_1 + (P_{12} - P_{22})x_2 = 0 \tag{36}$$

Since (36) is a linear equation in x_1 and x_2, we can solve for x_2 and substitute it into (35).

$$x_2 = mx_1 \qquad \text{where} \qquad m := \frac{P_{11} - P_{12}}{P_{22} - P_{12}}$$

$$\begin{aligned} \frac{\partial V}{\partial x} f(x) &= (P_{12}x_1 + P_{22}x_2)(-x_1 + \tfrac{1}{6}x_1^3) \\ &= (P_{12} + P_{22}m)(-x_1^2 + \tfrac{1}{6}x_1^4) \end{aligned} \tag{37}$$

The first term in (37) is a constant, which could be positive or negative, depending on the choice of P. The second term in (37) is a quartic function in x_1 and the roots of this function are $0, 0, -\sqrt{6}, \sqrt{6}$. A graph of the quartic function is shown in Figure 1. The interval $(-\sqrt{6}, \sqrt{6})$ is of opposite sign to $(-\infty, -\sqrt{6}) \bigcup (\sqrt{6}, \infty)$, regardless of how we chose the entries of P. As such, it is not possible for (37) to be negative definite, and hence this system is not globally stabilizable for a quadratic V. When the global CLF algorithm is used to search for a global CLF V of order 2, we get infeasible result, which is expected. The region where (37) is negative is on the line segment $x_2 = mx_1$, for

$x_1 \in (-\sqrt{6}, \sqrt{6})$ and $x_2 \in (-m\sqrt{6}, m\sqrt{6})$. Hence, one can enlarge this region by choosing a large m, which in turn implies a very narrow ellipse for the level set $\{x \in \mathbb{R}^2 | V(x) \leq 1\}$.

We use our algorithm to find a local CLF for this problem by setting the order of V, s_1, s_2, s_8 and p_4 to be 2, 4, 2, 2 and 5 respectively. We initialize V_0 to be a CLF of the linearized system: $V_0 = 7.636x_1^2 - 7.578x_1x_2 + 3.313x_2^2$. After 50 iterations, we get $\beta = 38.37$ and β is still monotonically increasing (See Figure 2). Figure 3 shows the level set $\{x \in \mathbb{R}^2 | V(x) \leq 1\}$ which is a region of attraction for this system when we use the resulting local CLF and the corresponding feedback law. The line segment shows the set $\left\{x \in \mathbb{R}^n \mid \frac{\partial V}{\partial x} g(x) = 0, \frac{\partial V}{\partial x} f(x) < 0\right\}$ and our region of attraction stays within this line segment in this direction.

5 Conclusion

In this paper, we presented our algorithms that search for CLFs for nonlinear systems that are affine in control and have polynomial vector fields. Traditionally, construction of CLFs requires intimate knowledge of the system involved before one can even propose a likely CLF candidate. Moreover, finding a CLF is often by trial-and-error. We hope that with our algorithms, finding CLFs for this class of nonlinear systems will be simplified and made systematic.

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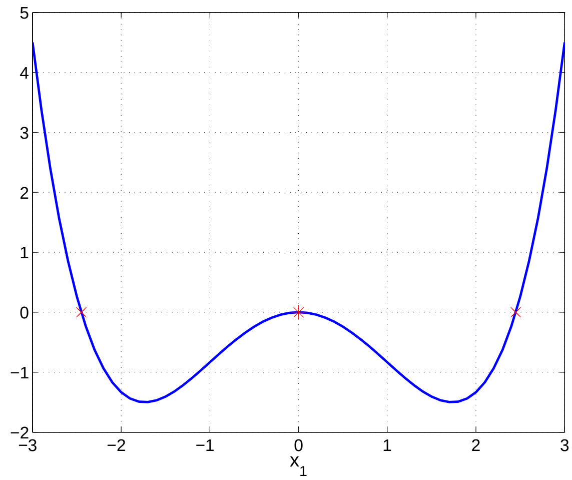

Figure 1: Graph of $-x_1^2 + \frac{1}{6}x_1^4$

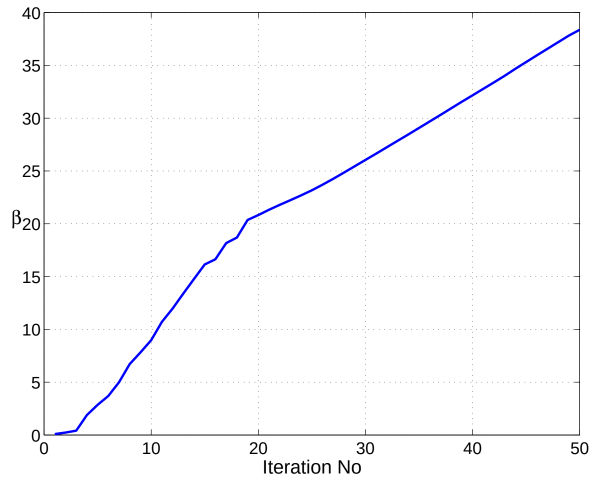

Figure 2: Progress of β during optimization.

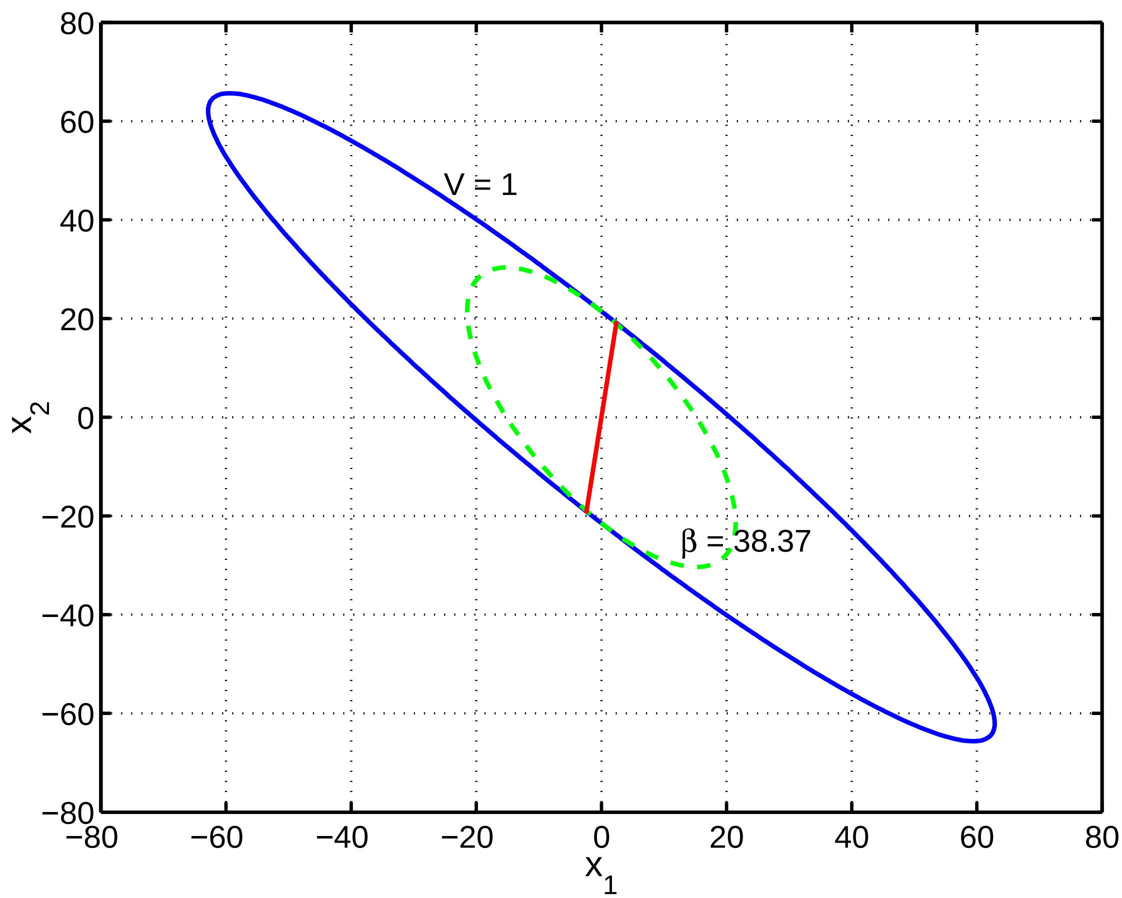

Figure 3: Closed loop system's region of attraction.