Nonlinear Control Analysis and Synthesis using Sum-of-Squares Programming

by

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Abstract

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This thesis considers Lyapunov based control analysis and synthesis methods for continuous time nonlinear systems with polynomial vector fields. We take an optimization approach of finding polynomial Lyapunov functions through the use of SOS programming and the application of the Positivstellensatz theorem.

There are three main areas considered in this thesis: local stability analysis, local performance analysis, and global and local controller and observer synthesis.

For local stability analysis, we present SOS programs that enlarge a provable region of attraction for polynomial systems. We propose using pointwise maximum and minimum of polynomials to reduce the number of decision variables and to obtain larger inner bounds on the region of attraction. This idea is illustrated most notably with a Van der Pol equations example. We also extend this region of attraction inner bound enlargement problem to polynomial systems with uncertain dynamics by considering both parameter-dependent and independent Lyapunov functions. Besides using the pointwise maximum of such functions, we also propose gridding the uncertain parameter space to further reduce the size of the SOS program. The significance of the gridding method is made apparent with two examples. A related stability region analysis problem of finding a tight outer bound for attractive invariant sets is also studied. We also present some computation statistics on a region of attraction benchmark example with arbitrary data and increasing problem size.

We study two local performance analysis problems for polynomial systems. The first is on finding outer bounds for the reachable set due to disturbances with $\mathcal{L}_2$ and $\mathcal{L}_\infty$ bounds. A SOS based refinement of the outer bound is proposed and illustrated with a previously studied example. The second problem is on finding an upper bound for the $\mathcal{L}_2 \to \mathcal{L}_2$ gain and its refinement. Interesting results are obtained when this method is applied to an adaptive control example.

For controller synthesis, we present SOS programs for finding global and local Control Lyapunov Functions. For observer synthesis, we formulate SOS programs that search for polynomial observers using Lyapunov based methods. Examples are provided to demonstrate these synthesis methods.

It is hoped that the optimization based methods in this thesis will complement existing nonlinear analysis and design methods.

Professor Andrew K. Packard
Dissertation Committee Chair

To my parents

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Notation

$\mathbb{R}$	The real numbers
$\mathbb{R}_+$	Non-negative real numbers
$\mathbb{R}^n$	Real n-vectors
$\mathbb{Z}$	The ring of integers
$\mathbb{Z}_+$	Non-negative integers
$M \succeq 0$	M is symmetric and positive semidefinite
$M \succ 0$	M is symmetric and positive definite
$\mathcal{R}_n$	The set of polynomials with real coefficients in n variables
Σ_n	The subset of $\mathcal{R}_n$ that are sum-of-squares polynomials

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Chapter 1

Introduction

In this thesis, we consider control analysis and synthesis problems for continuous time nonlinear systems with polynomial vector fields. The standard textbook approach to such problems for a general nonlinear system is to find a Lyapunov function that satisfies a list of conditions, from which certain properties of the system can be concluded. However, such an approach is generally presented as non-constructive and the choice of a candidate Lyapunov function is often by trial and error.

In addressing the non-constructive nature of finding a candidate Lyapunov function, we take the optimization route to search for the Lyapunov function. With the recent development of sum-of-squares (SOS) programming and the Positivstellensatz theorem from real algebraic geometry, we can try to pose control analysis and synthesis problems as SOS programs that are computationally tractable. In using SOS programs, we are restricting ourselves to systems with polynomial vector fields and polynomial Lyapunov functions. This class of problem can be extended to non-polynomial vector fields by recasting them into rational vector fields, at the expense of incurring additional variables and constraints. The recasting of non-polynomial vector fields is documented in [25] and is outside the scope of this thesis.

Even though a SOS program is computationally tractable, the problem size still grows rapidly as the number of variables and the degree of the polynomial grows. Hence, one emphasis of this thesis is on problem formulation that reduces the number of decision variables so that our method can be applied to larger problems.

We also emphasize local nonlinear analysis and synthesis problems, which are generally more useful in nonlinear systems. However, such local problems result in SOS programs that are bilinear in the decision polynomials. In our previous works [15], [34], algorithms were proposed that involved a two-way iterative search between the Lyapunov function and the SOS multipliers. With the recent introduction of YALMIP [23] and PENBMI [19], which allow for bilinear polynomial optimization, we can do away with the two-way iterative search, but as PENBMI is a local bilinear matrix inequality solver, convergence to the global optimum is not guaranteed.

1.1 Thesis Overview and Contributions

The outline of this thesis is as follows:

Chapter 2 gives the background material needed for problem formulation in the subsequent chapters. A brief overview of semidefinite programming, polynomial definitions, sum-of-squares programming, the Positivstellensatz theorem and its relation to the $\mathcal{S}$-procedure are presented. Of particular note is Section 2.3.3 on the computational aspects of sum-of-squares programming that will motivate the development of various methods in Chapter 3 to reduce the number of decision variables.

Chapter 3 addresses the local stability analysis of polynomial systems. In the first part, we consider the problem of enlarging a provable region of attraction for the polynomial system. The starting point is Section 4.2 of [14], where the search for a single polynomial Lyapunov function using SOS programming was proposed. This technique is extended in this thesis to the use of pointwise maximum and minimum of several polynomials. The advantage of using such composite Lyapunov functions is that composition of several low degree polynomials can often give level sets that are as rich as single Lyapunov functions of higher degrees, and at the same time utilizes fewer decision variables. Such a usage of composite Lyapunov functions is a recurring theme in this chapter. For systems without a local asymptotically stable equilibrium point, there might still be an invariant set, so a related problem of finding tight outer bounds for attractive invariant sets is studied. Next, we formulate SOS programs that enlarge a provable region of attraction for uncertain systems, using both parameter-dependent and independent Lyapunov functions. Besides the use of composition of such functions to reduce the number of decision variables, the method of solving the SOS program on a gridded uncertain parameter space is also proposed to further reduce the size of the SOS program. Lastly, we present some computation statistics on a region of attraction benchmark example with arbitrary data and increasing problem size to indicate how well our optimization methods and the bilinear solver perform.

Chapter 4 studies the local performance analysis of polynomial systems. The problem of finding an upper bound for the reachable set of a polynomial system that is subjected to an $\mathcal{L}_2$-norm bounded disturbance was first presented in [15]. In this thesis, we extend this idea further by formulating a SOS program based refinement on this upper bound. We also study a related problem of finding an upper bound for the reachable set due to disturbances with $\mathcal{L}_2$ and $\mathcal{L}_\infty$ bounds. The second part of this chapter derives an upper bound of the induced $\mathcal{L}_2 \to \mathcal{L}_2$ gain for a polynomial system. We also propose a refinement of this induced gain by using composite Lyapunov functions that are pointwise maximum of polynomials.

Chapter 5 covers controller and observer synthesis for polynomial systems. Unlike our previous work in [15], where a polynomial controller is explicitly synthesized, we take the approach of finding a Control Lyapunov Function (CLF) for the system using SOS programming, after which the control law is constructed from the CLF. We also present SOS programs that search for local CLFs which enlarge the closed loop system's region of attraction. For nonlinear observer synthesis, we take the approach of Lyapunov based methods, along the lines proposed by Vidyasagar [39].

Chapter 6 presents the conclusions and some recommendations of future research directions in this area.

In the **Appendix**, some practical aspects of using SOS programming are presented, based on this author's anecdotal experience. It is hoped that the reader finds this appendix useful when using SOS programming tools.

1.2 Summary of Examples

Listed below are the examples in this thesis.

1. Provable region of attraction enlargement:

   - Van der Pol equations: Section 3.1.4.1
   - Hahn's example: Section 3.1.4.2
   - 3-dimensional system with unstable limit cycle: Section 3.1.4.3

2. Attractive invariant sets:
    - Van der Pol oscillator: Section 3.2.1
3. Provable region of attraction enlargement for uncertain systems:
    - Uncertain Van der Pol equations: Section 3.3.4.1
    - Uncertain 3-dimensional system with unstable limit cycle: Section 3.3.4.2
4. Computation statistics:
    - Scalable benchmark example with known ROA: Section 3.4
5. Reachable set refinement:
    - CDC'03 example [15] revisited: Section 4.1.3
6. Upper bound of $\mathcal{L}_2$ to $\mathcal{L}_2$ gain:
    - Adaptive control example from Krstić [21]: Section 4.2.3
7. Control Lyapunov Functions:
    - Global stabilization of a bilinear system: Section 5.1.4.1
    - Global stabilization of a multi-input system: Section 5.1.4.2
    - Local stabilization of CDC'03 example [15]: Section 5.2.2
8. Nonlinear Observers:
    - Duffing equations: Section 5.3.3.1
    - CDC'03 example [15] using feedback of observed states: Section 5.3.3.2

Chapter 2

Background

In this chapter, we present a summary of the background material needed for problem formulation in subsequent chapters.

2.1 Semidefinite Programming

A *semidefinite program* (SDP) is a problem with a linear objective, and semidefinite constraints. Formally, suppose $c \in \mathbf{R}^m$, and $F_0, F_1, \ldots, F_m \in \mathbb{R}^{n \times n}$ are real, symmetric matrices. The SDP defined by them is

$$\min_{x \in \mathbf{R}^m} c^T x$$

subject to

$$F(x) := F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0 \tag{2.1}$$

Here, $\succeq$ means positive semidefinite (partial ordering for symmetric matrices). $F(x) \succeq 0$ is called a *linear matrix inequality* (LMI).

SDP has been studied extensively and good references on this topic include [38] and

[43]. The two most important properties of a SDP are that it is a convex optimization problem and it is computationally tractable. There are numerous SDP solvers available, such as SeDuMi [33], SDPT3 [36] and LMILAB. As these solvers have specific formats to describe SDPs, parsers such as YALMIP [23] and TKLMITOOL [9] are extremely useful in setting up SDPs in these specific formats.

2.2 Polynomial Definitions

Definition 2.1. *A* ***Monomial*** m_α *in* n *variables is a function defined as* $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ *for* $\alpha \in \mathbb{Z}_+^n$*. The degree of a monomial is defined as* $\deg m_\alpha := \sum_{i=1}^n \alpha_i$*.*

Definition 2.2. *A* ***Polynomial*** f *in* n *variables is a finite linear combination of monomials, with* $c_\alpha \in \mathbb{R}$*:*

$$f := \sum_\alpha c_\alpha m_\alpha = \sum_\alpha c_\alpha x^\alpha.$$

Define $\mathcal{R}_n$ to be the set of all polynomials in n variables. The degree of f is defined as $\deg f := \max_\alpha \deg m_\alpha$ (provided the associated c_α is non-zero).

Additionally, we define Σ_n to be the set of sum-of-squares (SOS) polynomials in n variables:

$$\Sigma_n := \left\{ p \in \mathcal{R}_n \;\middle|\; p = \sum_{i=1}^t f_i^2 \quad , \; f_i \in \mathcal{R}_n, \; i = 1, \ldots, t \right\}.$$

Obviously if $p \in \Sigma_n$, then $p(x) \geq 0 \; \forall x \in \mathbb{R}^n$. However, the converse not true, i.e. there are globally non-negative polynomials that are not SOS polynomials, as first noted by Hilbert [27].

2.3 Sum-of-squares Programming

In many control problems, there are conditions requiring a polynomial to be non-negative. However, checking whether a polynomial is globally non-negative is NP-hard when its degree is at least 4 [27], while checking whether a polynomial is SOS is a SDP, as we will show in the subsection below. As a consequence, conditions on non-negativity are replaced by sufficient conditions on the polynomial being SOS in our problem formulation in subsequent chapters. This replacement is often termed a “relaxation”.

2.3.1 Sum-of-squares Decomposition

Using a “Gram matrix” approach, Choi et al. [7] showed that given $p \in \mathcal{R}_n$ of degree $2d$, $p \in \Sigma_n$ if and only if $\exists\, Q \succeq 0$ such that

$$p(x) = z^T(x)Qz(x), \qquad z(x) = [1, x_1, x_2, \ldots, x_n, x_1x_2, \ldots, x_n^d]^T. \tag{2.2}$$

The vector $z(x)$ consists of $\binom{n+d}{d}$ monomials in x. Since the monomials are not algebraically independent, the matrix Q may not be unique, and there are some representations of p where $Q \succeq 0$, but not for others. By expanding $z^T(x)Qz(x)$ and matching the coefficients of x to polynomial p, we can show that the set of Q that satisfies (2.2) is an affine subspace.

If a $Q \succeq 0$ is found, by eigenvalue decomposition $Q = T^TDT$, where $D = diag\{d_i\}$, $d_i \geq 0$, and hence the SOS decomposition for $p(x) = \sum_i d_i(Tz)_i^2$. The number of squares of polynomials in the SOS decomposition of $p(x)$ is the same as the rank of Q.

Powers and Wörmann [28] proposed an algorithm to check if any $Q \succeq 0$ exists for a given $p \in \mathcal{R}_n$ using inefficient, but exact decision methods. However, they did not exploit the convex property of this problem, which Parrilo [27] showed is a SDP.

A simple example below illustrates the abovementioned points:

Let $p(x) := 1 + 3x_1^2 + 4x_1x_2^2 + x_1^4 + 4x_2^4$. To check if $p \in \Sigma_n$, let $p = z^T Q z$, where $z = [1, x_1, x_2, x_1^2, x_1x_2, x_2^2]^T$ and symmetric $Q \in \mathbb{R}^{6\times 6}$. If we can find a $Q \succeq 0$, then $p \in \Sigma_n$. By multiplying out $z^T(x)Qz(x)$ and matching the coefficients of x to polynomial p, we can see that there are 15 linear constraints in 21 variables as shown in the table below:

Monomials	Equalities
1	q_{11}=1
x_1	$2q_{12}$=0
x_2	$2q_{13}$=0
x_1^2	$2q_{14} + q_{22}$=3
x_1x_2	$2q_{15} + 2q_{23}$=0
x_2^2	$2q_{16} + q_{33}$=0
x_1^3	$2q_{24}$=0
$x_1^2x_2$	$2q_{25} + 2q_{34}$=0
$x_1x_2^2$	$2q_{26} + 2q_{35}$=4
x_2^3	$2q_{36}$=0
x_1^4	q_{44}=1
$x_1^3x_2$	$2q_{45}$=0
$x_1^2x_2^2$	$2q_{46} + q_{55}$=0
$x_1x_2^3$	$2q_{56}$=0
x_2^4	q_{66}=4

One particular choice of G_0 (not necessarily $\succeq 0$) such that $p = z^T G_0 z$ is

$$G_0 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

As the system of linear equations is under-determined, there are several degrees of freedom in finding the entries of Q. For example, the monomial x_1x_2 has the equality $2q_{15} + 2q_{23} = 0$. Since $2q_{15}(x_1x_2) - (x_1)2q_{23}(x_2) = 0$, adding any linear combinations of $2q_{15} - 2q_{23} = 0$ to $2q_{15} + 2q_{23} = 0$ still satisfies the constraint for the monomial x_1x_2. For this example, we can find 6 such relations that have $\{z^T G_i z\}_{i=1}^6 = 0$, which form an affine subspace for Q.

The search for a $Q \succeq 0$ such that $p = z^T Q z$ is precisely a SDP feasibility problem (2.1): Find $\lambda \in \mathbb{R}^6$ such that

$$Q = G_0 + \sum_{i=1}^{6} \lambda_i G_i \succeq 0$$

where

$$G_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad G_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$G_3 := \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad G_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_5 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad G_6 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $Q_0 := G_0$, $Q_1 := G_0 - 2G_6$ and $Q_2 := G_0 - G_3 - 2G_6$ are different representations of p. However $Q_0 \not\succeq 0$, but $Q_1 \succeq 0$ and $Q_2 \succeq 0$. Factoring, for example, Q_2 into $L^T L$, where

$$L := \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix},$$

we have the following SOS decomposition for $p(x)$:

$$p(x) = 1 + 3x_1^2 + 4x_1x_2^2 + x_1^4 + 4x_2^4 = (1 + x_1^2)^2 + (x_1 + 2x_2^2)^2.$$

2.3.2 Sum-of-squares Programming

Besides showing that the SOS decomposition problem is an SDP, Parrilo also proved the following simple extension:

Theorem 2.1 (Parrilo). *Given a finite set* $\{p_i\}_{i=0}^m \in \mathcal{R}_n$*, the existence of* $\{a_i\}_{i=1}^m \in \mathbb{R}$ *such that*

$$p_0 + \sum_{i=1}^{m} a_i p_i \in \Sigma_n$$

is an SDP feasibility problem.

This theorem is useful since it allows one to answer questions like the following SOS programming example.

Example 2.1. Given $p_0, p_1 \in \mathcal{R}_n$, does there exist a $k \in \mathcal{R}_n$, of a given degree, such that

$$p_0 + kp_1 \in \Sigma_n. \tag{2.3}$$

To answer this question, write k as a linear combination of its monomials $\{m_j\}$, $k = \sum_{j=1}^{s} a_j m_j$. Rewrite (2.3) using this decomposition

$$p_0 + kp_1 = p_0 + \sum_{j=1}^{s} a_j (m_j p_1). \tag{2.4}$$

Since $(m_j p_1) \in \mathcal{R}_n$, (2.4) is a feasibility problem that can be checked by Theorem 2.1.

SOS Programs

A software package, SOSTOOLS [29, 30], exists to aid in solving the LMIs that result from SOS programming. This package lets the user choose between using two SDP solvers SeDuMi [33] or SDPT3 [36]. A SOS program, as defined by SOSTOOLS, is of the form:

Given polynomials $\{a_{i,j}(x)\} \in \mathcal{R}_n$, search for $\{p_i(x)\}_{i=1}^{\hat{N}} \in \mathcal{R}_n$ and $\{p_i(x)\}_{i=\hat{N}+1}^{N} \in \Sigma_n$ that

$$\min \; w^T c$$

where c is a vector formed from the unknown coefficients of

polynomials $p_i(x)$ for $i = 1, 2, \ldots, \hat{N}$

SOS polynomials $p_i(x)$ for $i = (\hat{N}+1), \ldots, N$

such that

$$a_{0,j} + \sum_{i=1}^{N} p_i(x) a_{i,j}(x) = 0 \quad \text{for } j = 1, 2, \ldots, \widehat{J}$$

$$a_{0,j} + \sum_{i=1}^{N} p_i(x) a_{i,j}(x) \in \Sigma_n \quad \text{for } j = (\widehat{J}+1), \ldots, J$$

SOSTOOLS requires that the SOS program be linear in the decision polynomials. In the past [15, 16, 34], whenever we encountered SOS programs that were bilinear in the decision polynomials, we used “V-S” iteration, holding one set of decision polynomials fixed and optimizing over the other set (which is a SDP), then switching over the sets which we were optimizing and holding fixed.

More recently, YALMIP [23], a versatile parser, added SOS programming functionality that allows bilinear decision polynomials. This new development allows us to do away with the “V-S” iteration, but as it uses PENBMI [19], a local bilinear matrix inequality solver, convergence to the global optimum is not guaranteed.

2.3.3 Computational Aspects of SOS Programming

Despite having these software tools, we still run into dimensionality problems: the number of decision variables increases exponentially with the number of variables, n, and the degree of the polynomial, $2d$.

Recall in Section 2.3.1 that the set of $Q \succeq 0$ that satisfies $p(x) = z^T(x)Qz(x)$ is an affine subspace because the variables in z are not algebraically independent. The affine subspace is

$$\left\{ Q \in \mathbb{R}^{r \times r} \,\middle|\, Q = G_0 + \sum_{i=1}^{N_1} \lambda_i G_i \quad , \; \lambda_i \in \mathbb{R} \right\} \tag{2.5}$$

where

$$N_1 = \frac{1}{2}\left[\binom{n+d}{d}^2 + \binom{n+d}{d}\right] - \binom{n+2d}{2d}, \quad r = \binom{n+d}{d} \tag{2.6}$$

and the G_i's form a basis for Q. This representation of the affine subspace is known as explicit or image representation. Table 2.1 illustrates the exponential growth in the number of decision variables with n and $2d$: in each entry, the left column is r, while the right column is N_1. It is because of this exponential growth in the number of decision variables that motivates us to keep the degree of the polynomial as low as possible.

Table 2.1. r and N_1 wrt to n and $2d$

	$2d$									
n	2		4		6		8		10	
2	3	0	6	6	10	27	15	75	21	165
3	4	0	10	20	20	126	35	465	56	1310
4	5	0	15	50	35	420	70	1990	126	7000
6	7	0	28	196	84	2646	210	19152	462	98945
8	9	0	45	540	165	10692	495	109890	1287	785070
10	11	0	66	1210	286	33033	1001	457743	3003	4325750

We can also pose the SOS program in the dual form, and the affine subspace will be described by defining equations (also known as implicit or kernel representation) [27]. The advantage of this representation is that for high degree polynomials, the number of decision variables is significantly less than N_1. However, most of the problems we encounter are bilinear in the decision polynomials, which can be formulated easily in the explicit representation, but not easily in the implicit representation, so we cannot make use of the dual form.

2.4 The Positivstellensatz

In this section, we define concepts to state a central theorem from real algebraic geometry, the Positivstellensatz, which we will hereafter refer to as the P-satz. This is a powerful theorem which generalizes many known results. For example, applying the P-satz, it is possible to derive the $\mathcal{S}$-procedure by carefully picking the free parameters, as will be shown in section 2.4.1.

Definition 2.3. *Given $\{g_1, \ldots, g_t\} \in \mathcal{R}_n$, the **Multiplicative Monoid** generated by g_j's is the set of all finite products of g_j's, including 1 (i.e. the empty product). It is denoted as $\mathcal{M}(g_1, \ldots, g_t)$. For completeness define $\mathcal{M}(\phi) := 1$.*

An example: $\mathcal{M}(g_1, g_2) = \{g_1^{k_1} g_2^{k_2} \mid k_1, k_2 \in \mathbb{Z}_+\}$.

Definition 2.4. *Given $\{f_1, \ldots, f_r\} \in \mathcal{R}_n$, the **Cone** generated by f_i's is*

$$\mathcal{P}(f_1, \ldots, f_r) := \left\{ s_0 + \sum_{i=1}^{l} s_i b_i \;\middle|\; l \in \mathbb{Z}_+, s_i \in \Sigma_n, b_i \in \mathcal{M}(f_1, \ldots, f_r) \right\}.$$

Note that if $s \in \Sigma_n$ and $f \in \mathcal{R}_n$, then $f^2 s \in \Sigma_n$ as well. This allows us to express a cone of $\{f_1, \ldots, f_r\}$ as a sum of 2^r terms.

An example: $\mathcal{P}(f_1, f_2) = \{s_0 + s_1 f_1 + s_2 f_2 + s_3 f_1 f_2 \,|\, s_0, \ldots, s_3 \in \Sigma_n\}$.

Definition 2.5. *Given $\{h_1, \ldots, h_u\} \in \mathcal{R}_n$, the **Ideal** generated by h_k's is*

$$\mathcal{I}(h_1, \ldots, h_u) := \left\{ \sum h_k p_k \;\middle|\; p_k \in \mathcal{R}_n \right\}.$$

With these definitions we can state the following theorem, taken from [4, Theorem 4.2.2]:

Theorem 2.2 (Positivstellensatz). *Given polynomials $\{f_1, \ldots, f_r\}$, $\{g_1, \ldots, g_t\}$, and $\{h_1, \ldots, h_u\}$ in $\mathcal{R}_n$, the following are equivalent:*

1. The set below is empty:

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} f_1(x) \geq 0, \ldots, f_r(x) \geq 0, \\ g_1(x) \neq 0, \ldots, g_t(x) \neq 0, \\ h_1(x) = 0, \ldots, h_u(x) = 0 \end{array} \right\}$$

2. There exist polynomials $f \in \mathcal{P}(f_1, \ldots, f_r)$, $g \in \mathcal{M}(g_1, \ldots, g_t)$, $h \in \mathcal{I}(h_1, \ldots, h_u)$ such that

$$f + g^2 + h = 0.$$

In subsequent chapters, we will often encounter set containment questions of the form: Given $h, f_0, \ldots, f_r \in \mathcal{R}_n$, does the following set containment hold

$$\{x \,|\, h(x) = 0, f_1(x) \geq 0, \ldots, f_r(x) \geq 0\} \subseteq \{x \,|\, f_0(x) \geq 0\} \quad ? \tag{2.7}$$

The following proposition shows that the above set containment question can be posed as a SOS program with the application of P-satz and some simplifications.

Proposition 2.1. *If there exists $p \in \mathcal{R}_n$, $s_{01}, \ldots, s_{0r} \in \Sigma_n$ such that*

$$p(x)h(x) - \sum_{j=1}^{r} s_{0j}(x) f_j(x) + f_0(x) \in \Sigma_n. \tag{2.8}$$

then the set containment condition (2.7) holds.

Proof. Condition (2.7) is equivalent to

$$\{x \,|\, h(x) = 0, f_1(x) \geq 0, \ldots, f_r(x) \geq 0, -f_0(x) \geq 0, f_0(x) \neq 0\} = \emptyset. \tag{2.9}$$

Application of Theorem 2.2 (P-satz) to (2.9) means that (2.9) holds if and only if there exists $p \in \mathcal{R}_n$, $s_{(\cdot)} \in \Sigma_n$ and $k \in \mathbb{Z}_+$ such that

$$\hat{p}h + s + s_0(-f_0) + \sum_{i=1}^{r} s_i f_i + \sum_{j=1}^{r} s_{0j}(-f_0) f_j +$$
$$\sum_{i=1}^{r}\sum_{j=i}^{r} s_{ij} f_i f_j + \cdots + s_{0\ldots r}(-f_0)\prod_{i=1}^{r} f_i + f_0^{2k} = 0. \quad (2.10)$$

Setting $k = 1$, $\hat{p} = pf_0$, and all $s_{(\cdot)} = 0$ except $s_0, s_{01}, \ldots, s_{0r}$, we have sufficient condition

$$-f_0\left[-ph + s_0 + \sum_{j=1}^{r} s_{0j} f_j - f_0\right] = 0. \quad (2.11)$$

Since f_0 is not identically zero, the second term in (2.11) results in (2.8). □

2.4.1 Generalized $\mathcal{S}$-procedure

What does the $\mathcal{S}$-procedure [5] look like in the P-satz formalism? Given symmetric $n \times n$ matrices $\{A_k\}_{k=0}^{m}$, the $\mathcal{S}$-procedure states:

If there exist non-negative scalars $\{\lambda_k\}_{k=1}^{m}$ such that $A_0 - \sum_{k=1}^{m} \lambda_k A_k \succeq 0$, then

$$\bigcap_{k=1}^{m} \left\{x \in \mathbb{R}^n | x^T A_k x \geq 0\right\} \subseteq \left\{x \in \mathbb{R}^n | x^T A_0 x \geq 0\right\}.$$

Written in P-satz form, the question becomes

$$\text{“is } \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} x^T A_1 x \geq 0, \ldots, x^T A_m x \geq 0, \\ -x^T A_0 x \geq 0,\; x^T A_0 x \neq 0 \end{array} \right\} \text{ empty?”}$$

Certainly, if the λ_k exist, define $0 \preceq Q := A_0 - \sum_{k=1}^{m} \lambda_k A_k$. Further define SOS functions $s_0(x) := x^T Q x, s_{01} := \lambda_1, \ldots, s_{0m} := \lambda_m$. Note that

$$f := (-x^T A_0 x) s_0 + \sum_{k=1}^{m} (-x^T A_0 x)(x^T A_k x) s_{0k} \in \mathcal{P}\left(x^T A_1 x, \ldots, x^T A_m x, -x^T A_0 x\right)$$

and that $g := x^T A_0 x \in \mathcal{M}(x^T A_0 x)$. Substitution yields $f + g^2 = 0$ as desired.

The following lemma is a special case of the P-satz theorem and is a generalization of the $\mathcal{S}$-procedure. Instead of searching for non-negative scalars $\{\lambda_k\}_{k=1}^m$, we are searching over SOS polynomials $\{s_k\}_{i=1}^m$.

Lemma 2.1 (Generalized $\mathcal{S}$-procedure). *Given $\{p_i\}_{i=0}^m \in \mathcal{R}_n$. If there exist $\{s_k\}_{i=1}^m \in \Sigma_n$ such that $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, then*

$$\bigcap_{i=1}^m \{x \in \mathbb{R}^n \,|\, p_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \,|\, p_0(x) \geq 0\}.$$

Proof. Since $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, so $p_0 \geq \sum_{i=1}^m s_i p_i \; \forall x$. For any $\bar{x} \in \bigcap_{i=1}^m \{x \in \mathbb{R}^n \,|\, p_i(x) \geq 0\}$, since $s_i(\bar{x}) \geq 0$, so $\sum_{i=1}^m s_i p_i \geq 0$, hence $p_0(\bar{x}) \geq 0$. □

2.5 Remarks on Notation

We use the notation Σ_n to denote the set of SOS polynomials in n real variables. The particular variables are not noted, and usually there is an obvious n-dimensional variable present in the discussion. Similarly, the notation Σ_{n+m} also appears, meaning SOS polynomial in $n+m$ real variables, where, again, the particular variables are hopefully clear from the context of the discussion.

In several places, a relationship between an algebraic condition on some real variables and input/output/state properties of a dynamical system is claimed. In nearly all of these types of statements, we use same symbol for a particular real variable in the algebraic statement as well as the corresponding signal in the dynamical system. This could be a source of confusion, so care on the reader's part is required.

Chapter 3

Stability Analysis

Finding the stability region or region of attraction of a nonlinear system is a topic of significant importance and has been studied extensively, for example in [8], [11], [6] and [42]. It also has practical applications, such as determining the operating envelope of aircraft and power systems.

In this chapter, we present a method of using sum-of-squares (SOS) programming to search for polynomial Lyapunov functions that enlarge a provable region of attraction of nonlinear systems with polynomial vector fields. Lyapunov functions with degrees higher than quadratic have level sets that are richer than ellipses, and thus could potentially give larger provable regions of attraction.

A major problem with using higher degree Lyapunov functions is the extremely rapid increase in the number of optimization decision variables as the state dimension and the degree of the Lyapunov function (and the vector field) increase (see Section 2.3.3). As a result, in Section 3.1, we propose using pointwise maximum or minimum of polynomial functions to obtain richly shaped level sets while keeping the degree of polynomials low.

Figure 3.1 shows the level sets of pointwise maximum and minimum of two quadratic, positive definite functions. For a single Lyapunov function, only polynomials with degrees higher than quadratic would have such similar shapes for level sets.

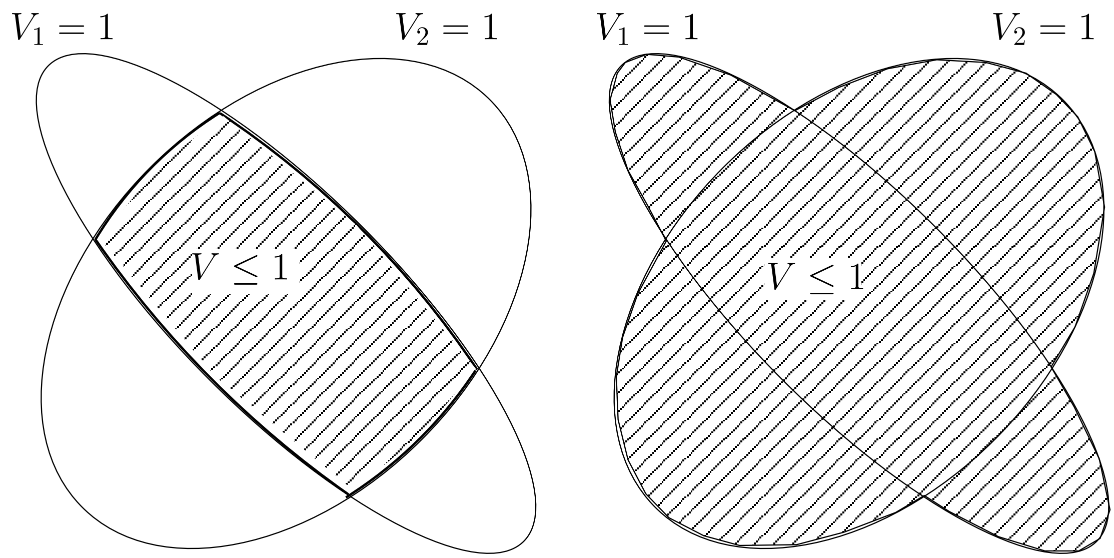

Figure 3.1. Left: $V = \max\{V_1, V_2\}$, Right: $V = \min\{V_1, V_2\}$

By its very nature, the region of attraction analysis is only applicable to systems with local asymptotically stable equilibrium points. For systems without such properties, we study a related stability region problem of finding a tight outer bound for an attractive invariant set in Section 3.2.

In Section 3.3, we consider the problem of enlarging a provable region of attraction for polynomial systems with uncertainty using both parameter-dependent and independent Lyapunov functions and the pointwise maximum of such functions.

Finally in Section 3.4, we present computation statistics of a benchmark example of provable region of attraction enlargement. The statistics are compiled to give us a idea of how well our optimization problems and the bilinear solver perform with arbitrary data and increasing problem size.

3.1 Enlarging Region of Attraction

Consider a system of the form

$$\dot{x} = f(x) \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ and f is a n-vector of elements of $\mathcal{R}_n$ with $f(0) = 0$. We want to find a provable region of attraction for this system, i.e. all points starting in this region will be attracted to the fixed point at the origin.

The following lemma on finding a region of attraction using Lyapunov function is a modification of a lemma from [40, pg. 167] and [18, pg. 122]:

Lemma 3.1. *If there exists a continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$V \text{ is positive definite,} \tag{3.2}$$

$$\Omega := \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\} \text{ is bounded, and} \tag{3.3}$$

$$\{x \in \mathbb{R}^n \,|\, V(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n \,|\, \tfrac{\partial V}{\partial x} f(x) < 0\} \tag{3.4}$$

then for all $x(0) \in \Omega$, *the solution of* (3.1) *exists and* $\lim_{t\to\infty} x(t) = 0$. *As such,* Ω *is a subset of the region of attraction for* (3.1).

Proof. Let $\Omega_r := \{x \in \mathbb{R}^n \,|\, V(x) \leq r \leq 1\}$, so $\Omega_r \subseteq \Omega$ and hence Ω_r is bounded. Because $\dot{V} < 0$ on $\Omega_r \setminus \{0\}$, if $x(0) \in \Omega_r$, $V(x(t)) \leq V(x(0)) \leq r$ while the solution exists. This means that solution starting inside Ω_r will remain in Ω_r while the solution exists. Since Ω_r is compact, the system (3.1) has an unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega_r$.

Take $\epsilon > 0$. Define the set $S_\epsilon := \{x \in \mathbb{R}^n \,|\, \frac{\epsilon}{2} \leq V(x) \leq 1\}$. Note that $S_\epsilon \subseteq \Omega \setminus \{0\} \subseteq \{x \in \mathbb{R}^n \,|\, \frac{\partial V}{\partial x} f(x) < 0\}$. Since S_ϵ is a compact set, $\exists\, r_\epsilon > 0$ such that $\dot{V} \leq -r_\epsilon < 0$ on S_ϵ. This implies that $\exists\, t^*$ such that $V(x(t)) < \epsilon$ for all $t > t^*$, i.e. $x(t) \in T_\epsilon := \{x \in \mathbb{R}^n \,|\, V(x) < \epsilon\}$ for all $t > t^*$. This shows that if $x(0) \in \Omega$, $V(x(t)) \to 0$ as $t \to \infty$.

Now, we need to show that $x(t) \to 0$ as $t \to \infty$ as well. Let $\epsilon > 0$. Define $\Omega_\epsilon := \{x \in \mathbb{R}^n \,|\, \|x\| \geq \epsilon, V(x) \leq 1\}$. Ω_ϵ is closed and bounded, with $0 \notin \Omega_\epsilon$. Since V is continuous and positive definite, and Ω_ϵ is compact, $\exists\, \gamma$ such that $V(x) \geq \gamma > 0$ on Ω_ϵ. We have already established that $V(x(t)) \to 0$ as $t \to \infty$, so $\exists\, \hat{t}$ such that for all $t > \hat{t}$, $V(x(t)) < \gamma$ and hence $x(t) \notin \Omega_\epsilon$, which means $\|x(t)\| < \epsilon$. So $x(t) \to 0$ as $t \to \infty$. $\square$

In order to enlarge Ω (by choice of V), we define a variable sized region $P_\beta := \{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\}$, and maximize β while imposing the constraint $P_\beta \subseteq \Omega$. Here, $p(x)$ is a positive definite polynomial, chosen to reflect the relative importance of the states. With the application of Lemma 3.1, the problem can be posed as the following optimization problem:

$$\max_{V \in \mathcal{R}_n} \beta \qquad \text{s.t.}$$

$$V(x) > 0 \text{ for all } x \in \mathbb{R}^n \backslash \{0\} \text{ and } V(0) = 0, \tag{3.5}$$

$$\text{the set } \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\} \text{ is bounded,}$$

$$\{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\}, \tag{3.6}$$

$$\{x \in \mathbb{R}^n \,|\, V(x) \leq 1\} \backslash \{0\} \subseteq \{x \in \mathbb{R}^n \,|\, \tfrac{\partial V}{\partial x} f(x) < 0\}. \tag{3.7}$$

3.1.1 Single Lyapunov Function

For a single differentiable Lyapunov function, [14, §4.2.2] presented the sufficient conditions for finding such a V. We will reproduce the steps leading to the sufficient conditions as the exposition for composite Lyapunov functions will be similar.

Expressing (3.5)–(3.7) as empty set conditions, we get:

$$\{x \in \mathbb{R}^n \,|\, V(x) \leq 0, x \neq 0\} = \emptyset, \tag{3.8}$$

$$\{x \in \mathbb{R}^n \,|\, p(x) \le \beta, V(x) \ge 1, V(x) \ne 1\} = \emptyset, \tag{3.9}$$

$$\{x \in \mathbb{R}^n \,|\, V(x) \le 1, \tfrac{\partial V}{\partial x} f(x) \ge 0, x \ne 0\} = \emptyset. \tag{3.10}$$

If $l_i(x)$ is a given positive definite polynomial, then the constraint $x \ne 0$ is equivalent to $l_i(x) \ne 0$. Usually, we take $l_i(x)$ of the form $l_i(x) = \sum_{j=1}^{n} \epsilon_{ij} x_j^2$, where ϵ_{ij} are positive numbers. Applying P-satz (Theorem 2.2) to each of the constraints (3.8)–(3.10), and using l_1 for (3.8) and l_2 for (3.10) respectively, we have

$$s_1 - V s_2 + l_1^{2k_1} = 0, \tag{3.11}$$

$$s_3 + (\beta - p)s_4 + (V-1)s_5 + (\beta - p)(V-1)s_6 + (V-1)^{2k_2} = 0, \tag{3.12}$$

$$s_7 + (1-V)s_8 + \tfrac{\partial V}{\partial x} f s_9 + (1-V)\tfrac{\partial V}{\partial x} f s_{10} + l_2^{2k_3} = 0 \tag{3.13}$$

where $s_1, \ldots, s_{10} \in \Sigma_n$ and $k_1, k_2, k_3 \in \mathbb{Z}_+$.

To keep the degree of the polynomial in each constraint low, and also to make the problem in the form solvable by SOS software, we simplify the constraints (3.11)–(3.13). This will result in sufficient conditions that are relaxations of the original problem.

First, pick $k_1 = k_2 = k_3 = 1$. Next, pick $s_2 = l_1$ and $s_1 = l_1 \hat{s}_1$ and factor out l_1 in (3.11). In (3.12), set $s_3 = s_4 = 0$ and factor out a $(V-1)$ term. For (3.13), set $s_{10} = 0$ and factor out l_2.

After these simplifications, we have

Optimization Problem 3.1 (Single Lyapunov ROA):

$$\max \beta \quad \text{over } V \in \mathcal{R}_n,\ V(0) = 0, \quad s_6, s_8, s_9 \in \Sigma_n,$$

such that

$$V - l_1 \in \Sigma_n, \tag{3.14}$$

$$-\Big((\beta - p)s_6 + (V-1)\Big) \in \Sigma_n, \tag{3.15}$$

$$-\big((1-V)s_8 + \tfrac{\partial V}{\partial x} f s_9 + l_2\big) \in \Sigma_n. \tag{3.16}$$

Note that in (3.14), V is underbounded by l_1, a positive definite function, so V is positive definite if constraint (3.14) is satisfied. Moreover, with V underbounded by l_1, the set $\Omega := \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\}$ is bounded. We can also obtain constraint (3.15) directly from (3.6) by using the generalized $\mathcal{S}$-procedure (Lemma 2.1). We call Ω a provable region of attraction because with the polynomials found from the above optimization, verifying (3.11) – (3.13) is straightforward.

3.1.2 Pointwise Max

Lemma 3.2. *If there exist continuously differentiable functions* $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$V := \max\{V_1, V_2\},\ V \text{ is positive definite,} \tag{3.17}$$

$$\Omega := \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\} \text{ is bounded,} \tag{3.18}$$

$$R_1 := \{x \in \mathbb{R}^n \,|\, V_2(x) \leq V_1(x) \leq 1\},\ R_1 \setminus \{0\} \subseteq \{x \in \mathbb{R}^n \,|\, \tfrac{\partial V_1}{\partial x} f(x) < 0\}, \tag{3.19}$$

$$R_2 := \{x \in \mathbb{R}^n \,|\, V_1(x) \leq V_2(x) \leq 1\},\ R_2 \setminus \{0\} \subseteq \{x \in \mathbb{R}^n \,|\, \tfrac{\partial V_2}{\partial x} f(x) < 0\} \tag{3.20}$$

then for all $x(0) \in \Omega$*, the solution of* (3.1) *exists and* $\lim_{t\to\infty} x(t) = 0$*. As such,* Ω *is a subset of the region of attraction for* (3.1).

Proof. Since $R_1 \cup R_2 = \Omega$, conditions (3.19) and (3.20) ensures that if $x(0) \in \Omega$, $V(x(t)) \leq V(x(0)) \leq 1$ while the solution exists. This means that solution starting inside Ω will remain in Ω while the solution exists. Since Ω is compact, the system (3.1) has an unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega$.

Take $\epsilon > 0$. Define $S_\epsilon := \{x \in \mathbb{R}^n \,|\, \frac{\epsilon}{2} \leq V(x) \leq 1\}$, so $S_\epsilon \subseteq (R_1 \cup R_2) \setminus \{0\}$. Note that for each i, $(S_\epsilon \cap R_i) \subseteq R_i \setminus \{0\} \subseteq \{x \in \mathbb{R}^n \,|\, \frac{\partial V_i}{\partial x} f(x) < 0\}$, so on the compact set $S_\epsilon \cap R_i$, $\exists\, r_{i,\epsilon}$, such that $\frac{\partial V_i}{\partial x} f(x) \leq -r_{i,\epsilon} < 0$. Consequently, if $x(t) \in S_\epsilon \cap R_1$ on $[t_A, t_B]$, then $V(x(t_B)) \leq -r_{1,\epsilon}(t_B - t_A) + V(x(t_A))$. Similarly, if $x(t) \in S_\epsilon \cap R_2$ on $[t_A, t_B]$, then $V(x(t_B)) \leq -r_{2,\epsilon}(t_B - t_A) + V(x(t_A))$. Therefore, if $x(t) \in S_\epsilon \cap (R_1 \cup R_2)$ on $[t_A, t_B]$, then $V(x(t_B)) \leq -r_\epsilon(t_B - t_A) + V(x(t_A))$, where $r_\epsilon = \min(r_{1,\epsilon}, r_{2,\epsilon})$. Since $r_\epsilon > 0$, this implies that $\exists\, t^*$ such that $V(x(t)) < \epsilon$ for all $t > t^*$, i.e. $x(t) \in T_\epsilon := \{x \in \mathbb{R}^n \,|\, V(x) < \epsilon\}$ for all $t > t^*$. This shows that if $x(0) \in \Omega$, $V(x(t)) \to 0$ as $t \to \infty$.

Since V is positive definite and continuous, and the set Ω is bounded, these conditions are exactly the same as those conditions used in showing that $x(t) \to 0$ as $t \to \infty$ in Lemma 3.1, so with $V(x(t)) \to 0$ as $t \to \infty$, $x(t) \to 0$ as well. □

Extension of Lemma 3.2 to $V(x) = \max_{i=1}^{q}\{V_i(x)\}$ for $q > 2$ (but finite) is obvious. Let $l_1(x)$ be a given positive definite polynomial. To simplify the SOS formulation for $V(x)$ being positive definite, we require that each V_i be underbounded by $l_1(x)$, so that each V_i is positive definite, i.e.

$$V_i - l_1 \in \Sigma_n, \quad \text{for } i = 1, \ldots, q. \tag{3.21}$$

Constraints (3.21) are sufficient conditions for V to be positive definite (3.17). Again, since V is underbounded by l_1, the set $\{x \in \mathbb{R}^n \,|\, V(x) \leq 1\}$ is bounded.

For constraint (3.6), note that $\{x \in \mathbb{R}^n \,|\, V(x) \leq 1\} = \cap_{i=1}^{q}\{x \in \mathbb{R}^n \,|\, V_i(x) \leq 1\}$, so

$$\begin{aligned} &\{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\} \\ \equiv &\{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\} \subseteq [\cap_{i=1}^{q}\{x \in \mathbb{R}^n \,|\, V_i(x) \leq 1\}] \\ \equiv &\{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n \,|\, V_i(x) \leq 1\}, \quad \text{for } i = 1, \ldots q. \end{aligned} \tag{3.22}$$

Conditions (3.19) – (3.20) are satisfied when each constraint below is satisfied:

$$
\begin{aligned}
&\{x \,|\, V_1(x) \le 1\}\backslash\{0\} \subseteq \{x \,|\, \tfrac{\partial V_1}{\partial x} f(x) < 0\} \qquad \text{when } V_1(x) \ge V_j(x),\ j = 2, \ldots, q \\
&\qquad\qquad\vdots \\
&\{x \,|\, V_q(x) \le 1\}\backslash\{0\} \subseteq \{x \,|\, \tfrac{\partial V_q}{\partial x} f(x) < 0\} \qquad \text{when } V_q(x) \ge V_j(x),\ j = 1, \ldots, q-1
\end{aligned}
\tag{3.23}
$$

Applying P-satz to the constraints (3.21) - (3.23) and making simplifying choices, we have the following sufficient conditions:

Optimization Problem 3.2 (Pointwise maximum ROA):

$$\max \beta \quad \text{over } V_i \in \mathcal{R}_n,\ V_i(0) = 0,\ s_{6i}, s_{8i}, s_{9i}, s_{0ij} \in \Sigma_n, \quad i = 1, \ldots q$$

such that

$$V_i - l_1 \in \Sigma_n, \tag{3.24}$$

$$-\Big((\beta - p)s_{6i} + (V_i - 1)\Big) \in \Sigma_n, \tag{3.25}$$

$$-\Big[(1 - V_i)s_{8i} + \tfrac{\partial V_i}{\partial x} f s_{9i} + l_2\Big] - \sum_{\substack{j=1 \\ j \ne i}}^{q} s_{0ij}(V_i - V_j) \in \Sigma_n. \tag{3.26}$$

There are $3q$ SOS constraints for this optimization problem.

3.1.3 Pointwise Min

Consider the case when $V(x) = \min_{i=1}^{q}\{V_i(x)\}$. Again, constraints (3.21) are sufficient conditions for V to be positive definite (3.5). We use the set containment condition $\{x \in \mathbb{R}^n \,|\, p(x) \le \beta\} \subseteq \{x \in \mathbb{R}^n \,|\, V(x) < 1\}$, which is a sufficient condition for constraint (3.6).

$$
\begin{aligned}
&\{x \in \mathbb{R}^n \,|\, p(x) \le \beta\} \subseteq \{x \in \mathbb{R}^n \,|\, V(x) < 1\} \\
\equiv&\{x \in \mathbb{R}^n \,|\, p(x) \le \beta\} \subseteq [\cup_{i=1}^{q} \{x \in \mathbb{R}^n \,|\, V_i(x) < 1\}] \\
\equiv&\{x \,|\, p(x) \le \beta\} \cap [\cup_{i=1}^{q} \{x \,|\, V_i(x) < 1\}]^c \quad \text{is empty}
\end{aligned}
$$

$$\equiv \{x \mid p(x) \le \beta\} \cap [\cap_{i=1}^{q} \{x \mid V_i(x) \ge 1\}] \quad \text{is empty} \tag{3.27}$$

Apply P-satz to (3.27), and removing the higher order terms in the cone, we have the following sufficient condition for (3.27):

$$-\left[s_{10}(\beta - p) + \sum_{i=1}^{q} s_{1i}(V_i - 1) + 1\right] \in \Sigma_n \tag{3.28}$$

The development for constraint (3.7) is similar to the case when $V = \max_{i=1}^{q}\{V_i\}$ and will not be repeated here. Hence, sufficient conditions for finding a provable region of attraction for $V = \min_{i=1}^{q}\{V_i\}$ are:

Optimization Problem 3.3 (Pointwise minimum ROA):

$$\max \beta \quad \text{over } s_{10}, s_{1i}, s_{8i}, s_{9i}, s_{0ij} \in \Sigma_n, \quad V_i \in \mathcal{R}_n,\ V_i(0) = 0, \quad i = 1, \ldots, q$$

such that

$$V_i - l_1 \in \Sigma_n, \tag{3.29}$$

$$-\left[s_{10}(\beta - p) + \sum_{i=1}^{q} s_{1i}(V_i - 1) + 1\right] \in \Sigma_n, \tag{3.30}$$

$$-\left((1 - V_i)s_{8i} + \tfrac{\partial V_i}{\partial x} f s_{9i} + l_2\right) - \sum_{\substack{j=1 \\ j \neq i}}^{q} s_{0ij}(V_j - V_i) \in \Sigma_n. \tag{3.31}$$

There are $2q + 1$ SOS constraints for this optimization problem.

3.1.4 Examples

In this subsection, we present three examples to illustrate our methods of enlarging provable regions of attraction. The first two examples are two state systems, where their exact stability boundaries are known and can be easily shown on phase portraits. We shall benchmark inner bounds on the region of attraction obtained with our methods against the exact stability boundaries. The third example is a three state system whose exact stability boundary is not known. Here, it is certainly more convenient to describe inner bounds on region of attraction for this system using level sets of Lyapunov functions.

3.1.4.1 Example 1 - Van der Pol equations

The system is

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2. \end{aligned} \tag{3.32}$$

It has an unstable limit cycle and a stable origin. The problem of finding its region of attraction have been studied extensively, for example, in [8, 11, 6]. More recently, [24] uses SOS programming and polynomial Lyapunov functions to find a provable region of attraction for this system. The region of attraction for this system is the region enclosed by its limit cycle, which can be easily obtained from the numerical solution of the ODE. However, our goal is to use Lyapunov functions. Initially, p is chosen to be

$$p(x) = x^T \begin{bmatrix} 3.78e\text{-}01 & -1.37e\text{-}01 \\ -1.37e\text{-}01 & 2.78e\text{-}01 \end{bmatrix} x \ , \tag{3.33}$$

whose level set at $\beta = 1$ is an ellipsoid that almost touches the limit cycle (see Figure 3.2).

The results of optimization problem 3.1 for 2nd, 4th and 6th degree single Lyapunov functions are listed in Table 3.1. Figure 3.3 shows the limit cycle and the level sets of the

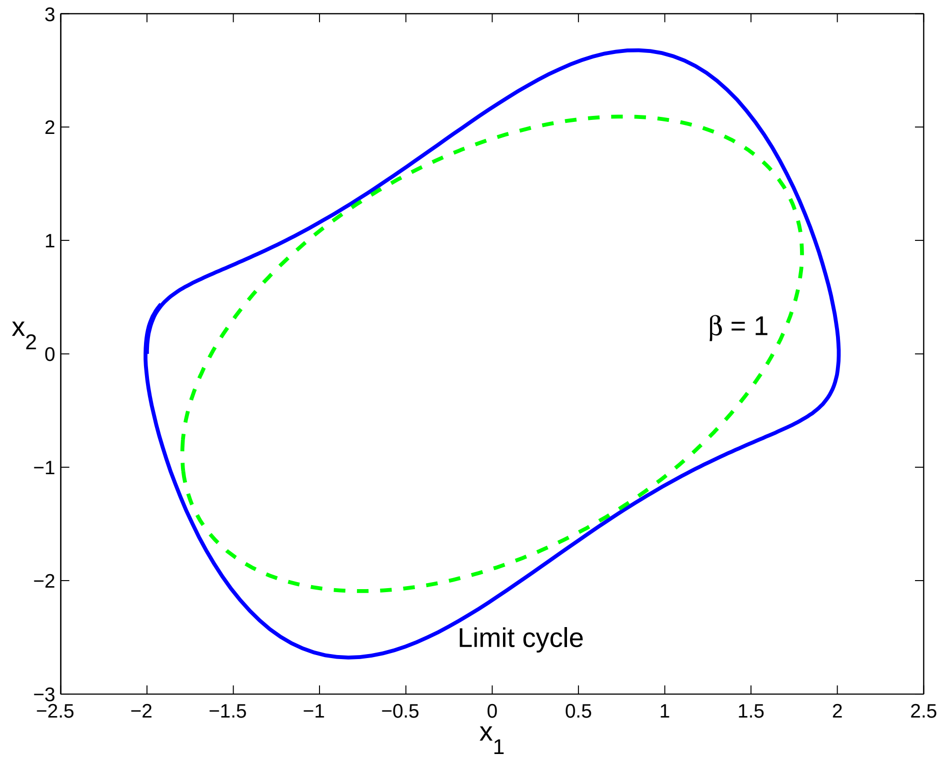

Figure 3.2. VDP: Limit cycle and level set $p(x) = 1$

certifying Lyapunov functions. The dashed ellipse is the level set $p(x) = \beta$, but only for $\beta = 0.91$. With the exception of degree 4, these results compare favorably with the results from [24]. For 8th and 10th degree single Lyapunov functions, we were unsuccessful in getting a β that is larger than from the 6th degree V. This is likely due to the non-convex nature of the problem, namely constraint (3.16), which is made worse with the increase in the number of variables for high degree V's. Since PENBMI is a local BMI solver, the optimization generally returns a local maxima instead of the global maxima.

Table 3.1. VDP: Single Lyapunov function

degree of					total no. of
V	s_6	s_8	s_9	β	decision variables
2	0	2	0	0.593	13
4	2	2	0	0.659	57
6	4	2	0	0.909	166
8	6	2	0	0.695	392
10	8	2	0	0.833	795

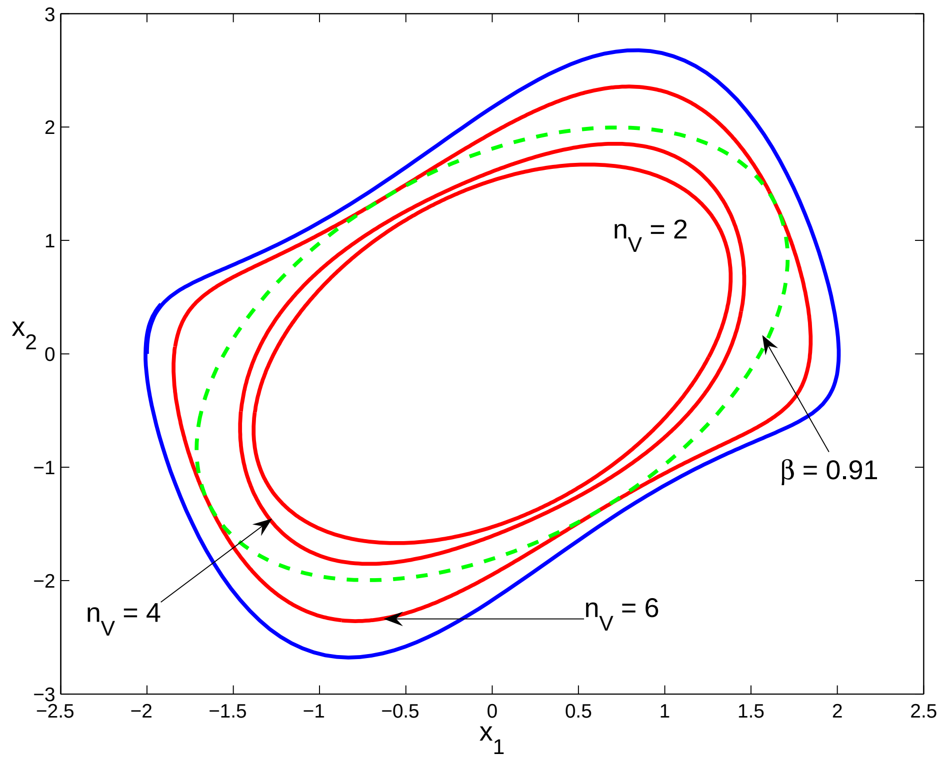

Figure 3.3. VDP: Provable ROA using single Lyapunov functions

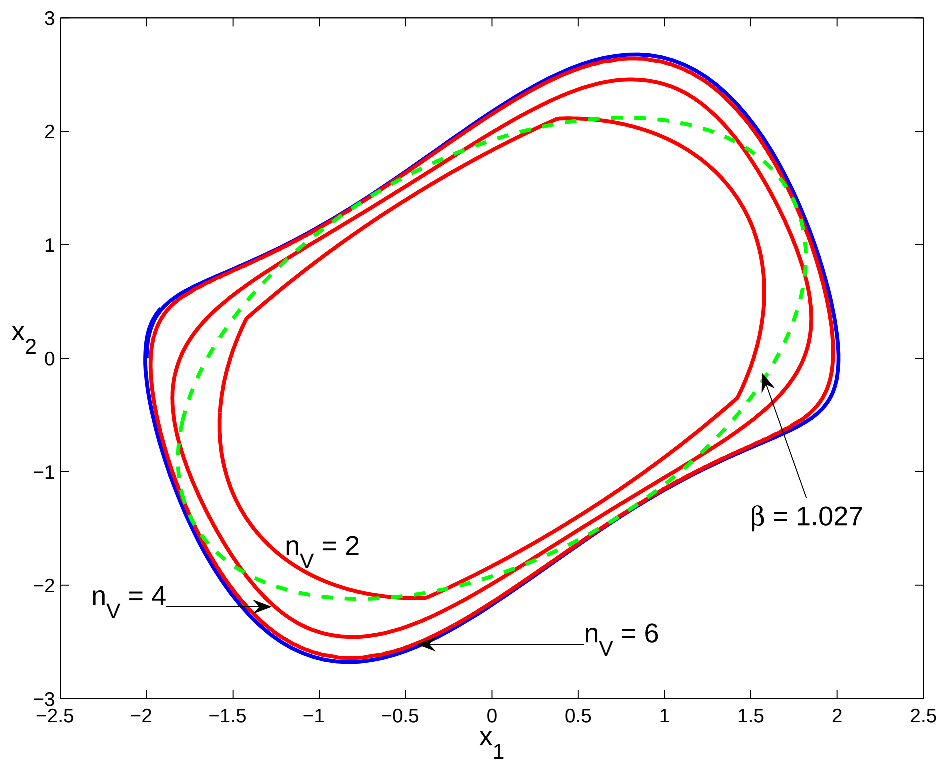

Figure 3.4. VDP: Provable ROA using pointwise max of two polynomials

Table 3.2. VDP: $\max_{i=1}^{q}\{V_i\}$

	degree of						total no. of
q	V	s_{6i}	s_{8i}	s_{9i}	s_{0ij}	β	decision variables
2	2	0	2	0	2	0.754	38
2	4	2	2	0	2	0.928	120
2	6	4	2	0	2	1.027	338
3	2	0	2	0	2	0.754	73
3	2	0	4	2	4	0.824	265

On the other hand, if we use optimization problem 3.2 to search for the pointwise maximum of two fixed degree V's, we get much improved results (see Table 3.2). For example, with two quartic V's we get slightly better β than a single 6th degree V (0.928 vs 0.909), with the advantage of fewer decision variables (120 vs 166). For the same degree, the pointwise maximum of two V's consistently gives much better result than a single V.

As we can see in Figure 3.4, the level set of pointwise maximum of two 6th degree V's almost approaches the limit cycle. These results compare very favorably with [6] and [24], and compared with the latter, require fewer number of decision variables.

In the pointwise maximum of two 6th degree V's, it is interesting to observe how the two V's interact. Figure 3.5 shows that for V_1, its level set (solid red lines) is disjointed - there is a set inside the limit cycle and two "islands" outside the limit cycle. Inside the "islands", $\dot{V}_1 \not< 0$, but the "islands" are excluded by V_2 (dashed green line) as $V_2 > V_1$ in those places. As a result, the composite level set of $\{x \,|\, \max\{V_1, V_2\} \leq 1\}$ satisfies the conditions in Lemma 3.1, but the set $\{x \,|\, V_1(x) \leq 1\}$ does not. Since the shape of the level set $\{x \,|\, V_2(x) \leq 1\}$ looks like the limit cycle, it might be tempting to assume that this level set is a region of attraction. Figure 3.6 refutes this assumption – inside the level set, points that are $\dot{V}_2 < 0$ are plotted as ($\bullet$), while points that are $\dot{V}_2 \geq 0$ are plotted as ($+$). As we can see, $\dot{V}_2$ is not negative everywhere inside $\{x \,|\, V_2(x) \leq 1\} \setminus \{0\}$, so the level is a not a region of attraction. In the pointwise maximum of V_1 and V_2, V_2 plays the role of a

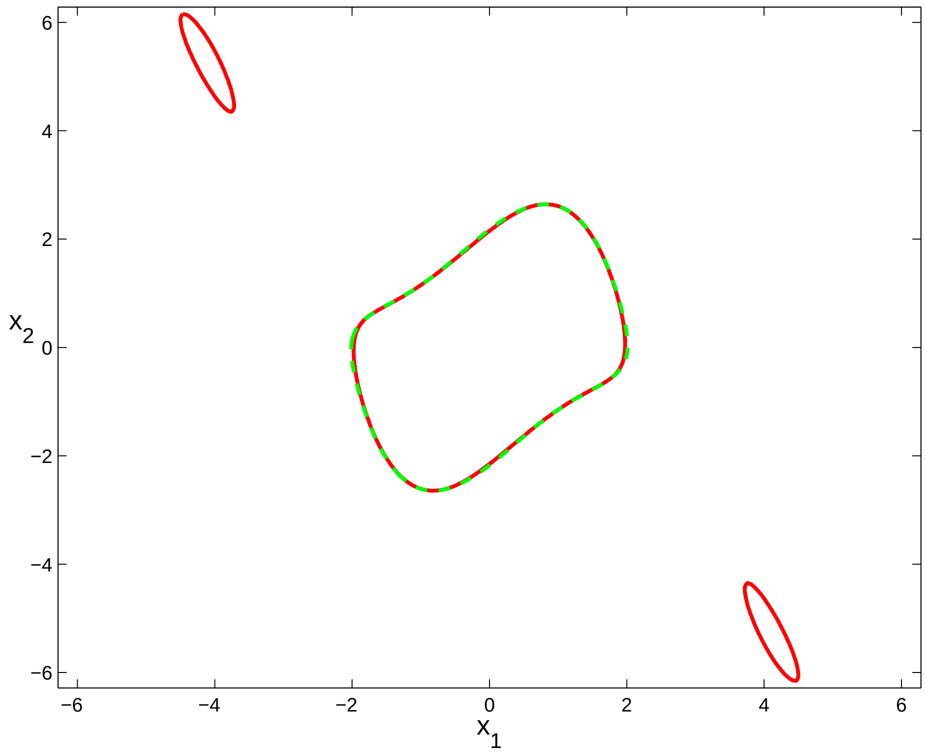

Figure 3.5. VDP: Level sets of two 6th degree polynomials at $V_1, V_2 = 1$

barrier function to eliminate the "islands" of V_1. This is further illustrated in Figure 3.7, which shows the level sets of V_1 and V_2 for different values of the level set. For V_1 and V_2 less than 0.525, $V_1 > V_2$, so the level set of V consists entirely of the level set of V_1 only. At $V_1 \approx 0.525$, the "islands" start appearing and their size grow as the level set value is increased. These "islands" are excluded because $V_2 > V_1$ in these regions.

Although the pointwise maximum of two polynomials yields much better results using less decision variables, the pointwise maximum of three polynomials does not have the same benefits. For the same degrees of the SOS multipliers, the pointwise maximum of three quadratic polynomials only yields the same result as the pointwise maximum of two quadratic polynomials (see the first row and the 2nd last row of Table 3.2). Closer examination of the three polynomials reveals that one of the polynomial is redundant, e.g. it is either the same as one of the other two or its level set is much bigger than the other two,

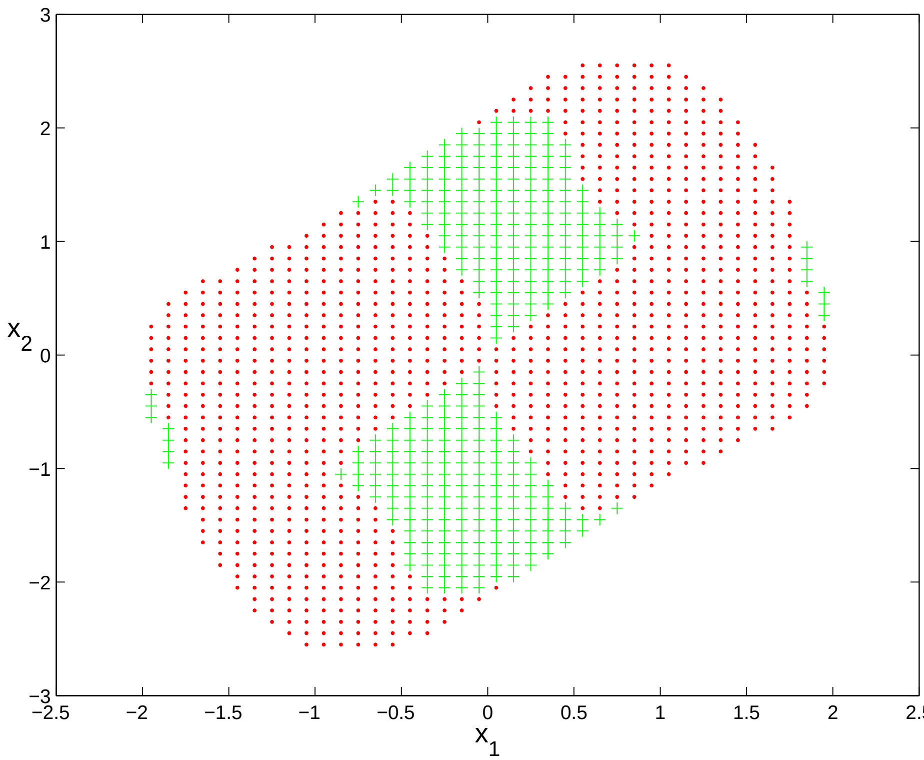

Figure 3.6. VDP: Inside $V_2 \leq 1$, points where $\dot{V}_2 < 0$ (•) and $\dot{V}_2 \geq 0$ (+)

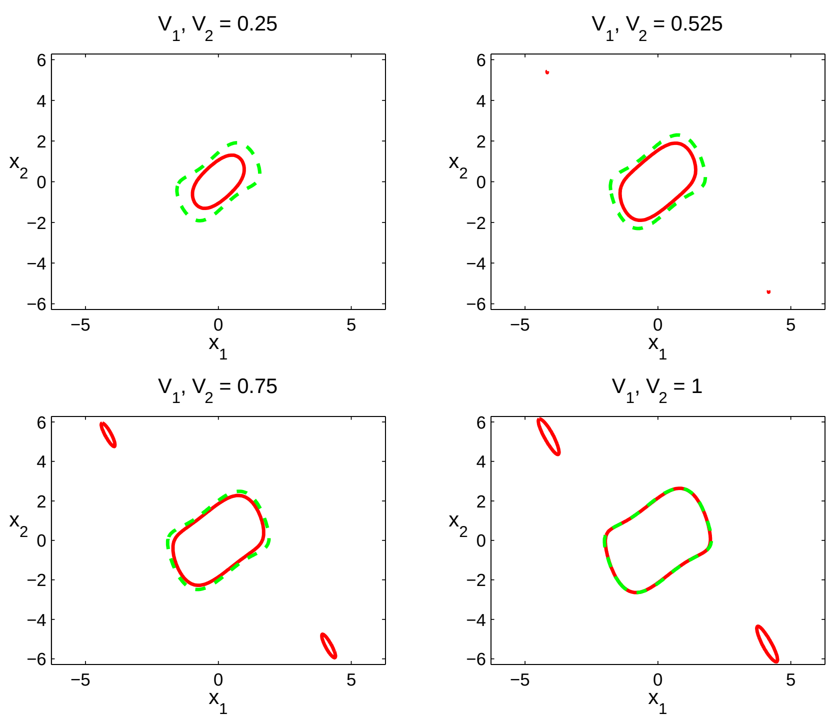

Figure 3.7. VDP: Level sets of two 6th degree polynomials

i.e. no part of its level set is used in the pointwise maximum. To get better results for pointwise maximum of three quadratic polynomials, we have to increase the degrees of the SOS multipliers, however, this increases the number of decision variables as well (see the last row of Table 3.2). For this case, we obtained a larger β value than the pointwise maximum of two quadratic polynomials, but this value is smaller than the pointwise maximum of two quartic polynomials and it uses about twice as many decision variables. Figure 3.8 shows provable ROA for pointwise maximum of two and three polynomials.

The p in (3.33) is chosen so that when $\beta = 1$, the set $\{x \,|\, p(x) \leq 1\}$ is a good approximation of the limit cycle. As such, it is not surprising that when $\beta \approx 1$ is obtained, the corresponding provable ROA approaches the limit cycle. What if another $p(x)$ is used? Suppose p is chosen to be

$$p_2(x) = x^T \begin{bmatrix} 2.78e\text{-}01 & 1.37e\text{-}01 \\ 1.37e\text{-}01 & 3.78e\text{-}01 \end{bmatrix} x. \tag{3.34}$$

We tried to fit the largest level set of $p_2(x)$ into the level set of the pointwise maximum of two quadratic V's using p, and we get $\beta = 0.323$. When we re-run the optimization (3.24) – (3.26) for a new pointwise maximum of two quadratic V's using p_2, we get $\beta = 0.347$. This shows that for any chosen shape factor (p or p_2), the optimization is able to find the best solution and that p is not specially chosen to give good result for our optimization.

Figure 3.9 shows provable ROAs using p_2 and pointwise maximum of two polynomials of degrees 2, 4 and 6. For the degree 6 V's, the optimization pushes its level set against the limit cycle, which is unexpected as the corresponding level set of p_2 is much smaller than the limit cycle.

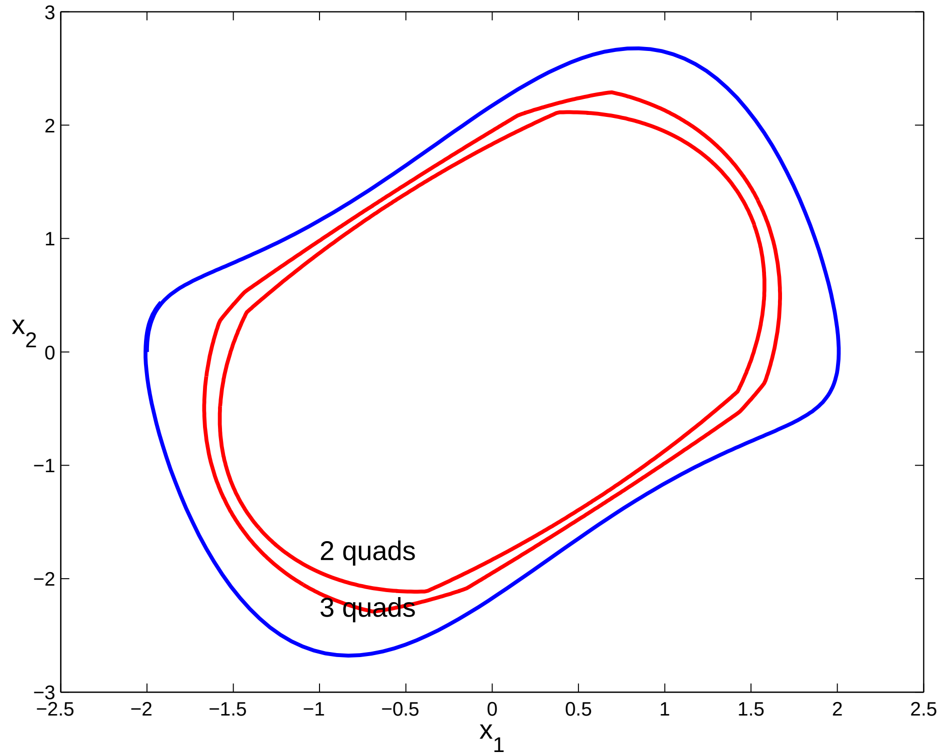

Figure 3.8. VDP: Provable ROA using pointwise max of 2 and 3 quadratic polynomials

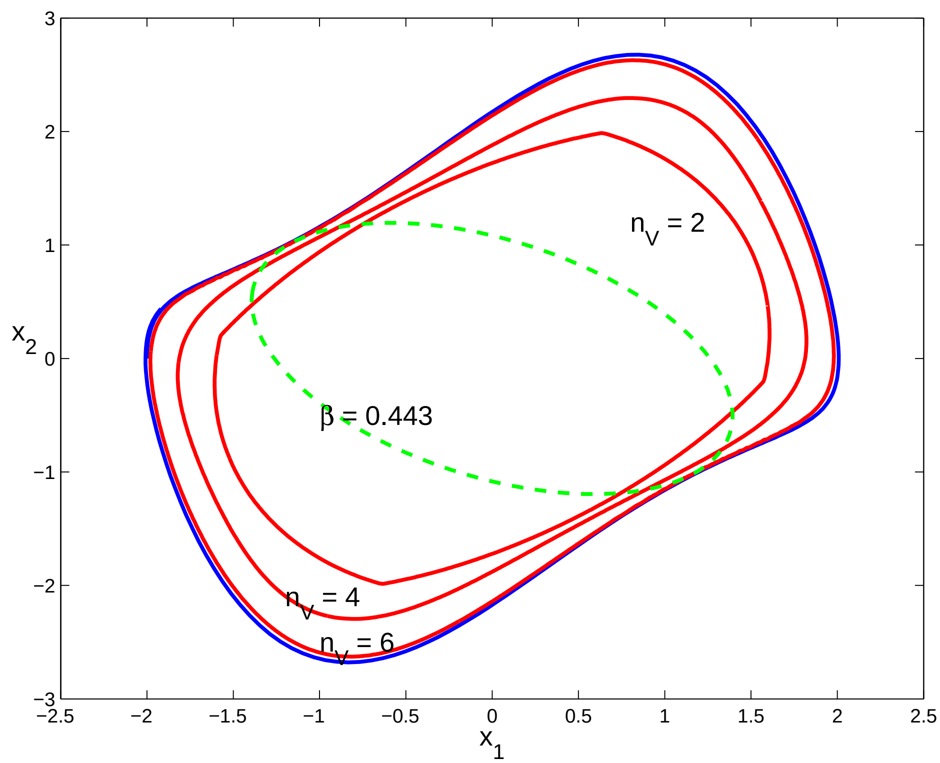

Figure 3.9. VDP: Provable ROA using pointwise max of two polynomials with $p_2(x)$

3.1.4.2 Example 2 - Hahn's example

This example is from Hahn [12], and is studied in [8, 11, 6].

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_1^2 x_2 \\ \dot{x}_2 &= -x_2. \end{aligned} \tag{3.35}$$

This system has an asymptotically stable origin and its exact stability region is known to be $x_1 x_2 < 1$ using Zubov's method, which involves solving partial differential equations. The Lyapunov function obtained using Zubov's method has exponential and rational terms: $V(x) = -1 + \exp\left(-\frac{x_2^2}{2} - \frac{x_1^2}{2(1-x_1 x_2)}\right)$.

We are interested to see how the results of optimization problems 3.1 and 3.2 using polynomials compare with the exact stability boundary. For this example, p is chosen to be

$$p(x) = x^T \begin{bmatrix} 14.47 & 18.55 \\ 18.55 & 26.53 \end{bmatrix} x. \tag{3.36}$$

The contour of $p = 10$ and the stability boundary $x_1 x_2 = 1$ is shown in Figure 3.10.

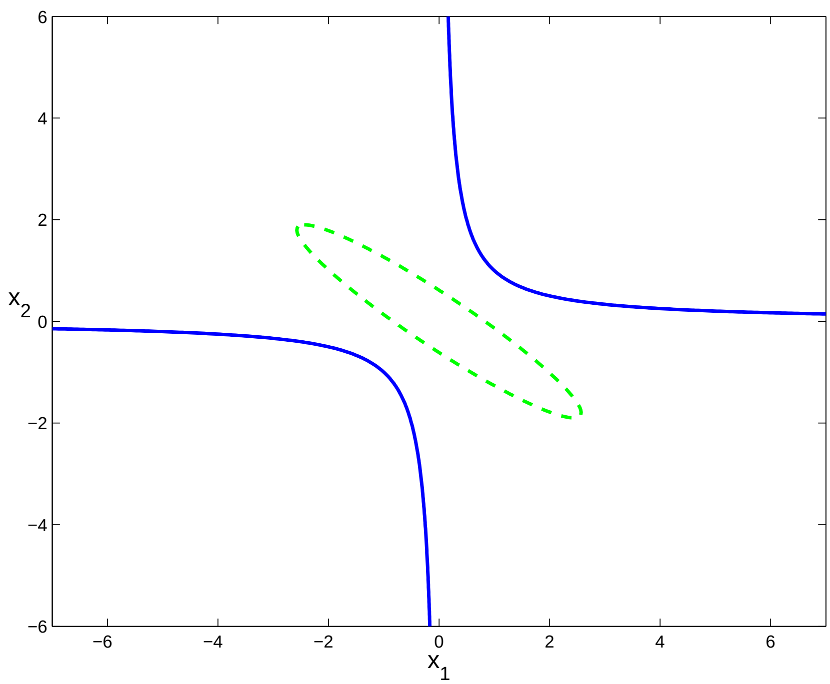

Figure 3.10. Hahn's example: Stability boundary $x_1 x_2 = 1$ and level set $p = 10$

We attempted optimization problems 3.1 and 3.2 using polynomial V's of degrees 2, 4 and 6. The results are shown in Table 3.3. In terms of β, the pointwise maximum of two quadratic V's easily beats a single V of degrees 2, 4 and 6. The pointwise maximum of two 6th degree V's achieves a β value that is 3 times larger than a single quadratic V. Since p is quadratic, this translates to $\sqrt{3} \approx 1.73$ or 73% larger in any direction of p's level set.

Table 3.3. Hahn's example

	degree of						
q	V	s_{6i}	s_{8i}	s_{9i}	s_{0ij}	β	total no. of decision variables
1	2	0	2	0	-	10.1	13
1	4	2	2	0	-	13.0	57
1	6	4	2	0	-	20.4	166
2	2	0	2	0	2	23.3	38
2	4	2	2	0	2	24.3	120
2	6	4	2	0	2	30.4	338

Figures 3.11 and 3.12 show provable regions of attraction using our method and for comparison purposes, the dashed-dotted ellipse is a region of attraction obtained by Davison [8] using numerical methods to search for a quadratic V. From the figures, we can see that we are able to enlarge provable regions of attraction when using higher degree Lyapunov functions and pointwise maximum of polynomials.

For that particular choice of p in (3.36), even though we have managed to enlarge provable regions of attraction, there are still regions near the origin that are not covered by these provable regions of attraction. To cover the remaining regions as close to the stability region $x_1x_2 < 1$ as possible, we shall use multiple p's. First, we start with

$$p_0(x) = x^T \begin{bmatrix} 40 & 0 \\ 0 & 1 \end{bmatrix} x. \tag{3.37}$$

and generate multiple p's by rotating p_0 every 3 degrees from 0 to 180 degrees. For each p, we run optimization problems 3.1 and 3.2 using single quartic Lyapunov functions and the pointwise maximum of two quartic polynomials.

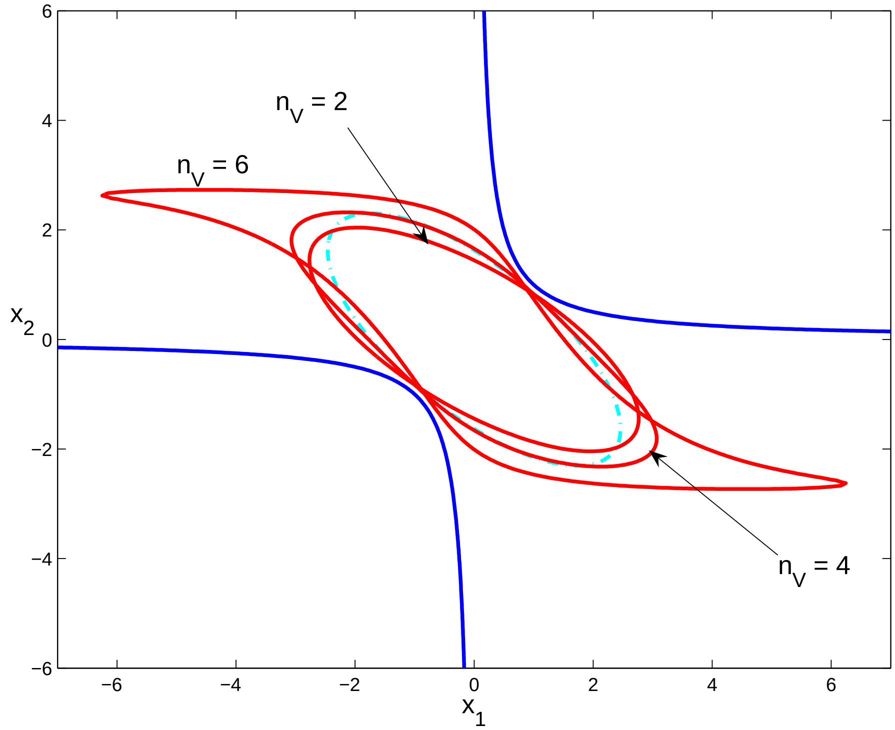

Figure 3.11. Hahn's example: Provable ROA using single Lyapunov functions

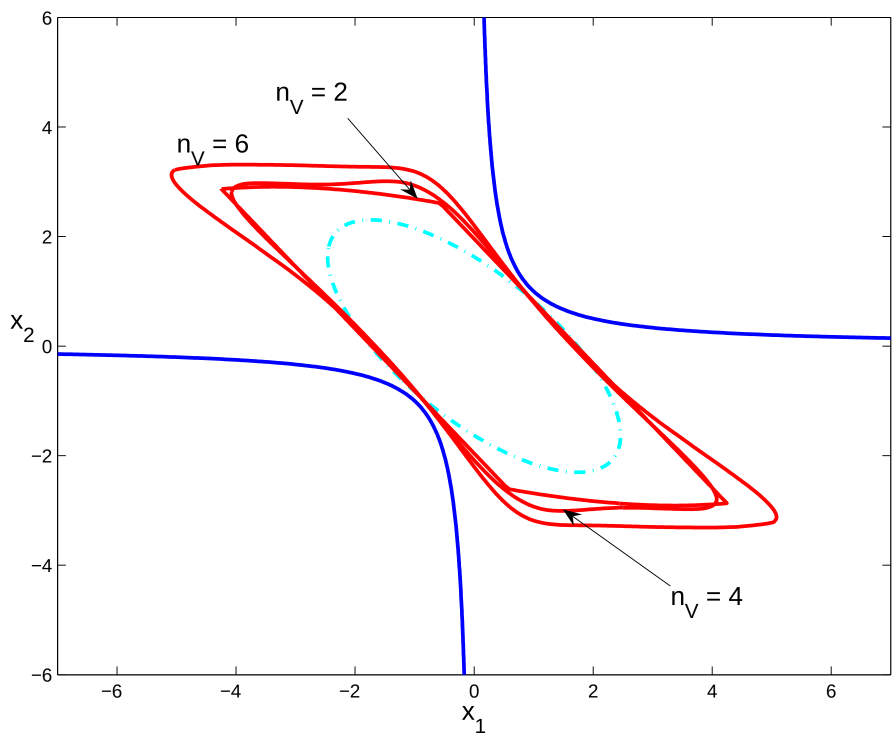

Figure 3.12. Hahn's example: Provable ROA using pointwise max of two polynomials

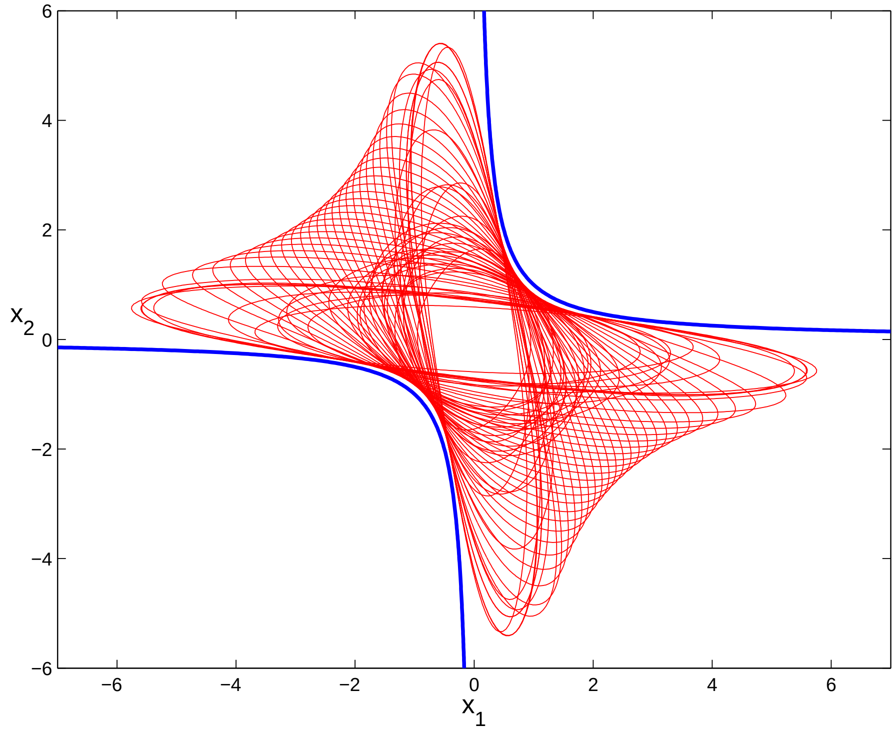

Figure 3.13. Hahn's example: Multiple ROAs using single Lyapunov functions

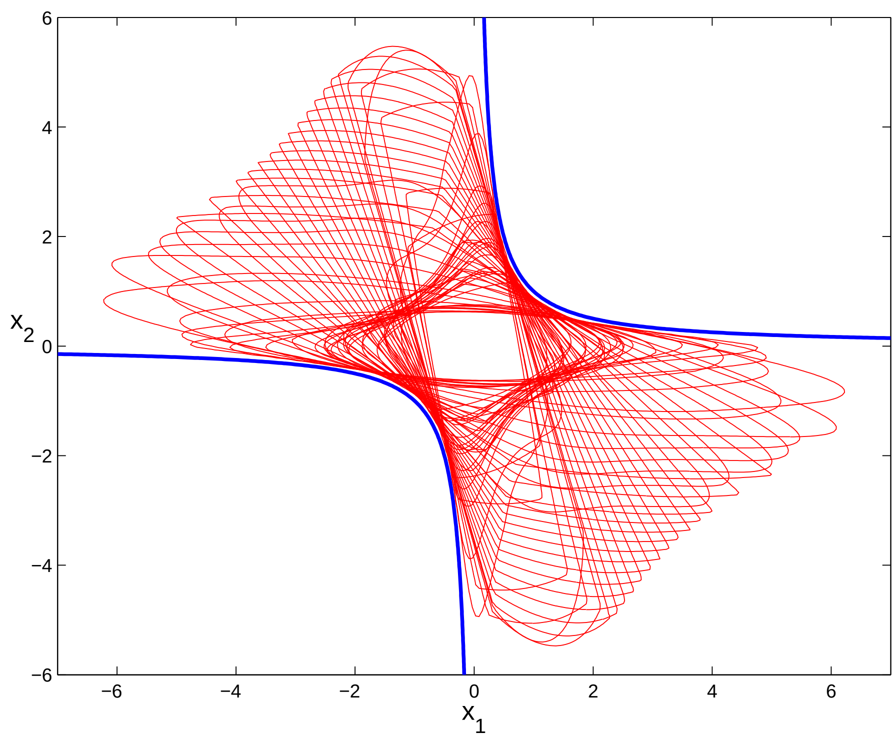

Figure 3.14. Hahn's example: Multiple ROAs using pointwise max of two polynomials

Figures 3.13 and 3.14 show the results obtained from multiple p's. Each closed curve in the figures corresponds to a provable region of attraction obtained by a particular p. We are able to cover larger stability regions using multiple p's: any point starting inside the union of all such regions will converge to the origin, albeit using different Lyapunov functions to prove asymptotic stability.

3.1.4.3 Example 3 - 3D example

This example is taken from [22] and has been studied in [8] and [6]. The system has a 3-dimensional unstable limit cycle and an asymptotically stable origin.

$$\begin{aligned}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= -x_3 \\
\dot{x}_3 &= -0.915x_1 + (1 - 0.915x_1^2)x_2 - x_3
\end{aligned} \tag{3.38}$$

The largest provable region of attraction obtained with methods developed in Davison [8], using a quadratic Lyapunov function, is $\beta = 1.0$, with

$$p(x) := x^T \begin{bmatrix} 12.5 & -8.1 & 3.0 \\ -8.1 & 20.8 & -8.5 \\ 3.0 & -8.5 & 13.4 \end{bmatrix} x. \tag{3.39}$$

We shall use the same $p(x)$ to enlarge a provable region of attraction using optimization problem 3.1 and 3.2. Interestingly, using a single quadratic Lyapunov function, we obtain $\beta = 2.76$. Pointwise maximum of two quadratic V's yields $\beta = 3.16$. Using a single quartic Lyapunov function, we obtained $\beta = 6.32$. Figure 3.15 shows a slice (the $x_2 - x_3$ plane at $x_1 = 0$) of this provable region of attraction, i.e. all points starting inside the region $\{x \,|\, p(x) \leq 6.32\}$ will converge to the origin. The two stars on the plot, which are just outside this region, are initial conditions whose trajectories diverge away from the origin.

Table 3.4. 3D Example

	degree of					p (3.39)	p_2 (3.40)	total no. of
q	V	s_{6i}	s_{8i}	s_{9i}	s_{0ij}	β	β	decision variables
1	2	0	2	0	-	2.76	8.16	25
2	2	0	2	0	2	3.16	9.19	81
1	4	2	2	0	-	6.32	14.59	200
2	4	2	2	0	2	6.41	16.17	412

These two points show that with respect to enlarging the estimate of the region of attraction in this direction (defined by p), the result is not conservative.

From Figure 3.15, the level set of that single quartic V indicates that it is approaching the stability boundary, so it would be difficult for the pointwise maximum of two quartic V's to improve upon set. Using optimization problem 3.2, the pointwise maximum of two quartic V's yield only a marginally larger β of 6.41. For illustrative purposes only, if we pick another p, such as

$$p_2(x) := x^T \begin{bmatrix} 33.1 & -28.6 & 4.88 \\ -28.6 & 56.7 & -29.2 \\ 4.88 & -29.2 & 36.7 \end{bmatrix} x, \tag{3.40}$$

we might be able to see the improvement from using composite Lyapunov functions. Table 3.4 summarizes the degrees of the polynomial used and the results from using p and p_2. With p_2, the pointwise maximum of two quartic V's show some improvement over single V. Figure 3.16 shows a slice (the $x_2 - x_3$ plane at $x_1 = 0$) of this provable region of attraction with p_2. From the figure, it might appear that the level set of $p_2 = 16.17$ does not touch the level set of pointwise max of two quartic V's, but in fact these two level sets do touch, but it cannot be seen in this sectional view.

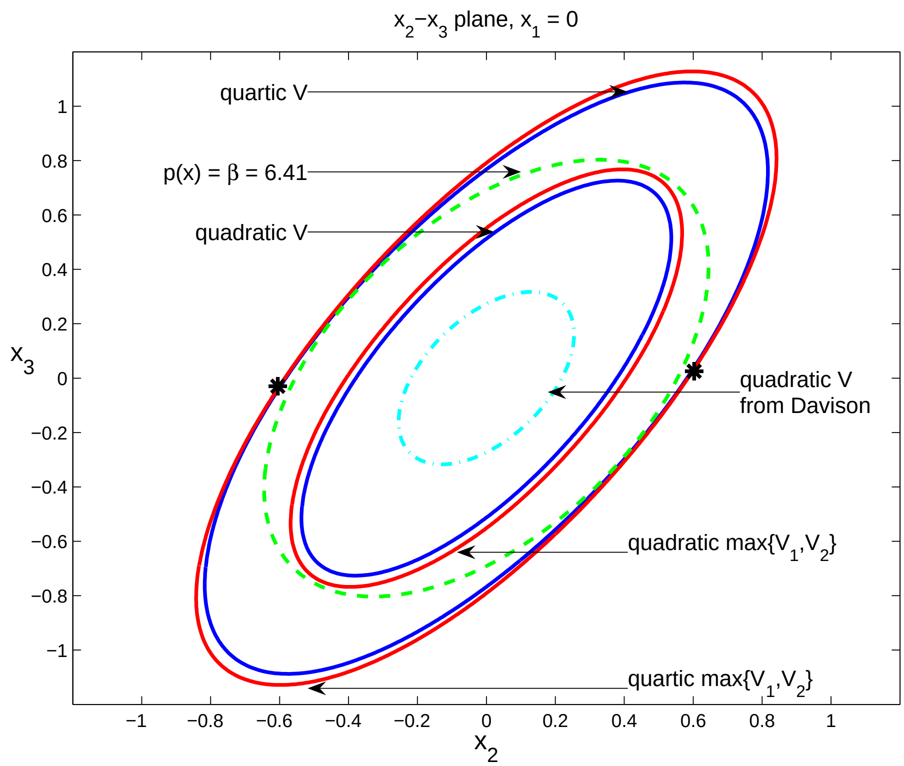

Figure 3.15. 3D Example: Provable ROA on $x_2 - x_3$ plane, $x_1 = 0$

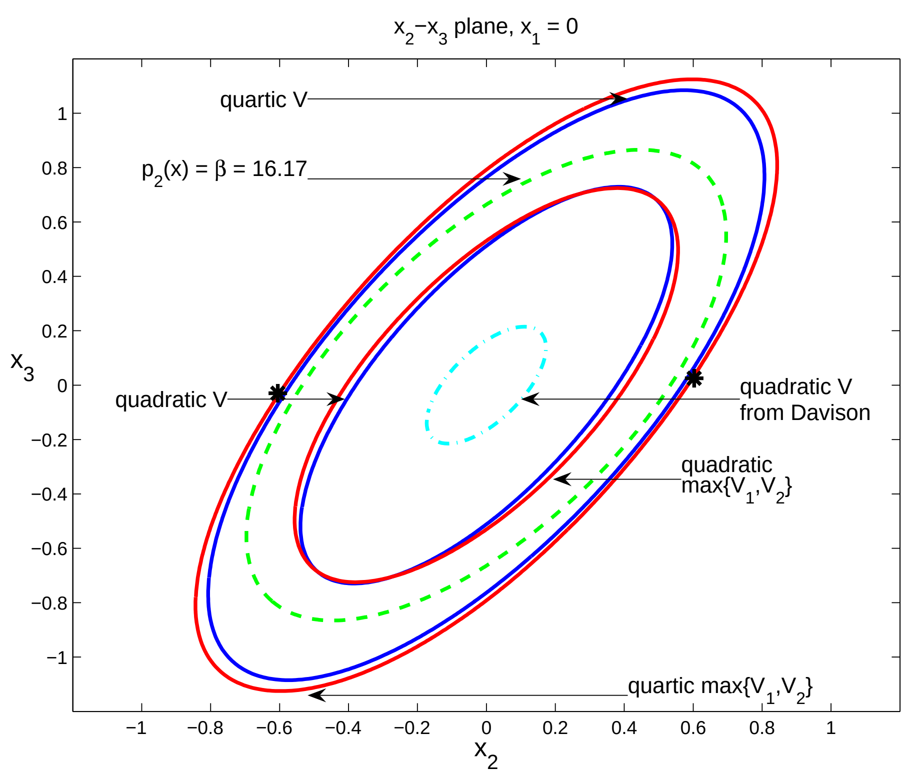

Figure 3.16. 3D Example: Provable ROA with $p_2(x)$ on $x_2 - x_3$ plane, $x_1 = 0$

3.2 Attractive Invariant Sets

For a system whose equilibrium point is not asymptotically stable, there might still be an attractive invariant set containing the equilibrium point, i.e. for points starting inside the set, their trajectories will remain in the set, and for points starting outside the set, their trajectories will eventually enter this set and remain inside. We are interested in finding a tight outer bound for this attractive invariant set, $\mathcal{I}$. Again, we introduce a variable sized region P_β such that $\mathcal{I} \subseteq P_\beta$. By minimizing β, we are tightening the outer bound for $\mathcal{I}$.

The problem of finding an outer bound of an attractive invariant set for system (3.1) can be posed as the following optimization problem:

Lemma 3.3. *If there exists $V \in \mathcal{R}_n$, $\epsilon > 0$ and $\beta > 0$ such that the following optimization is feasible*

$$\min_{V \in \mathcal{R}_n, \epsilon > 0} \quad \beta$$

$$V(x) > 0 \; \forall x \in \mathbb{R}^n \backslash \{0\},\; V(0) = 0, \text{ and radially unbounded,} \tag{3.41}$$

$$\{x \in \mathbb{R}^n \,|\, V \leq 1\} \subseteq \{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\} \tag{3.42}$$

$$\{x \in \mathbb{R}^n \,|\, V(x) \geq 1\} \subseteq \{x \in \mathbb{R}^n \,|\, \tfrac{\partial V}{\partial x} f(x) \leq -\epsilon\}\,, \tag{3.43}$$

then $\mathcal{I} := \{x \in \mathbb{R}^n \,|\, V \leq 1\}$ is an attractive invariant set for (3.1)*, and is contained in the set $P_\beta := \{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\}$.*

Proof. The derivative of V along the trajectory, $\dot{V} = \frac{\partial V}{\partial x} f(x)$, is strictly negative on the boundary of $\mathcal{I}$, so all trajectories starting inside $\mathcal{I}$ can never leave it, which makes $\mathcal{I}$ an invariant set. To show that a solution starting outside $\mathcal{I}$ eventually enters $\mathcal{I}$ in a finite time, suppose $V(x(0)) = c$, with $c > 1$. Note that $\dot{V} \leq -\epsilon < 0$ on the set $\{x \in \mathbb{R}^n \,|\, V(x) \geq 1\}$, so

$V(x(t)) \leq V(x(0)) - \epsilon t = c - \epsilon t$. As such, $V(x(t))$ will decay to the value 1 within the time interval $(0, (c-1)/\epsilon]$.

Moreover, since V is positive definite and radially unbounded by (3.41), the level sets $\{x \in \mathbb{R}^n \,|\, V \leq c\}$ for all $c \geq 1$ are invariant and bounded, so condition (3.43) guarantees that the solution trajectory remains bounded and exists for all time, and $V(x(t))$ eventually becomes ≤ 1. $\square$

We shall present the SOS formulation for the case of $V(x) = \min_{i=1}^{q}\{V_i(x)\}$, and omit the case for the pointwise maximum as it is similar.

Let $l_1(x)$ be a positive definite polynomial, and applying generalized $\mathcal{S}$-procedure to each of the constraints (3.41)–(3.43), we have the following sufficient conditions for the finding a tight outer bound for an attractive invariant set:

Optimization Problem 3.4 (Attractive Invariant Set):

$$\min \beta \quad \text{over } V_i \in \mathcal{R}_n,\ V_i(0) = 0, \quad s_{1i}, s_{2i}, s_{0ij} \in \Sigma_n,$$

$$\epsilon > 0, \qquad\qquad i = 1, \dots q$$

such that

$$V_i - l_1 \in \Sigma_n, \tag{3.44}$$

$$(\beta - p) - s_{1i}(1 - V_i) \in \Sigma_n, \tag{3.45}$$

$$-\epsilon - \frac{\partial V_i}{\partial x} f - s_{2i}(V_i - 1) - \sum_{\substack{j=1 \\ j \neq i}}^{q} s_{0ij}(V_j - V_i) \in \Sigma_n. \tag{3.46}$$

There are $3q$ SOS constraints for this optimization problem. Note that the above presentation is equally valid for a single V.

3.2.1 Example

The system is a Van der Pol oscillator with a stable limit cycle.

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1^2) - x_2 \\ \dot{x}_2 &= x_2(1 - x_2^2) + x_1. \end{aligned} \tag{3.47}$$

We want to find a tight outer bound on the attractive invariant set using optimization problem 3.4 and $p(x) = x_1^2 + x_2^2$. Table 3.5 shows the degree of the decision polynomials and the bound achieved for various V's. Figure 3.17 shows the limit cycle and the boundaries of the derived attractive invariant sets. For a single quartic V, this boundary is very close to the stable limit cycle.

Table 3.5. Attractive Invariant Set

	degree of					total no. of
q	V	s_{0ij}	s_{1i}	s_{2i}	β	decision variables
1	2	-	0	2	2.000	19
2	2	2	0	2	1.833	47
1	4	-	0	2	1.634	63

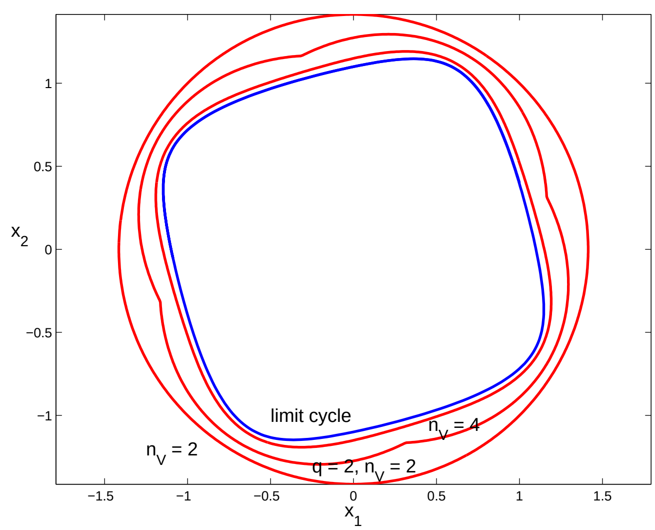

Figure 3.17. Attractive invariant sets for various V's

As a side note, we also attempt to find an attractive invariant set for the Van der Pol oscillator with a stable limit cycle, of the form similar to (3.32), but we were unsuccessful.

3.3 Enlarging a provable ROA for Uncertain Systems

Consider a nonlinear system with uncertain dynamics

$$\dot{x} = f(x, \delta) \tag{3.48}$$

and an apriori constraint on uncertainty: $N(\delta) \geq 0$. f is a n-vector of elements of $\mathcal{R}_{n+n_\delta}$, i.e. f is a n-vector of polynomials in (x, δ). We restrict our attention to f such that the equilibrium point $\bar{x} = 0$ that does not depend on δ, i.e.

$$f(\bar{x}, \delta) = 0 \quad \forall \delta \text{ such that } N(\delta) \geq 0.$$

We want to search for a parameter-dependent Lyapunov function $V(x, \delta)$ to maximize β so that for each δ satisfying $N(\delta) \geq 0$,

$$V(\cdot, \delta) \text{ is positive definite,} \tag{3.49}$$

$$\text{the set } \{x \,|\, V(x, \delta) \leq 1\} \text{ is bounded,}$$

$$\{x \,|\, p(x) \leq \beta\} \subseteq \{x \,|\, V(x, \delta) \leq 1\}, \tag{3.50}$$

$$\{x \,|\, x \neq 0, V(x, \delta) \leq 1\} \subseteq \{x \,|\, \tfrac{\partial V}{\partial x} f(x, \delta) < 0\}. \tag{3.51}$$

Note that for each fixed $\bar{\delta}$ such that $N(\bar{\delta}) \geq 0$, the above conditions (3.49) – (3.51) are exactly the same as (3.5) – (3.7) which are a direct application of Lemma 3.1. Hence, the set $\{x \,|\, V(x, \bar{\delta}) \leq 1\}$ is a region of attraction (ROA) for the system (3.48) with that particular $\bar{\delta}$.

The following lemma is useful in rewriting the conditions (3.49) – (3.51) with the qualifier "for each δ satisfying $N(\delta) \geq 0$" into plain set containment conditions.

Lemma 3.4. *For each $\bar{y}$ satisfying $f_3(\bar{y}) \leq 0$,*

$$A_1 := \{x \,|\, f_1(x, \bar{y}) \leq 0\} \subseteq \{x \,|\, f_2(x, \bar{y}) \leq 0\} =: A_2 \tag{3.52}$$

iff

$$B_1 := \{(x,y)\,|\,f_1(x,y) \le 0, f_3(y) \le 0\} \subseteq \{(x,y)\,|\,f_2(x,y) \le 0\} =: B_2 \tag{3.53}$$

Proof. ($\Rightarrow$) Let $(\bar{x},\bar{y}) \in B_1$, hence $f_3(\bar{y}) \le 0$ and $f_1(\bar{x},\bar{y}) \le 0$. Obviously $\bar{y}$ satisfies the hypothesis of (3.52), hence $x \in A_1$, so $x \in A_2$, i.e. $f_2(x,y) \le 0$, so $(x,y) \in B_2$.

($\Leftarrow$) Take any $\bar{y}$ such that $f_3(\bar{y}) \le 0$. Let $x \in A_1$, so $f_1(x,\bar{y}) \le 0$. $(x,\bar{y}) \in B_1 \Rightarrow (x,\bar{y}) \in B_2$, so $f_2(x,\bar{y}) \le 0$. Hence $x \in A_2$. □

3.3.1 Single Parameter-Dependent Lyapunov Function

If we restrict ourselves to searching for polynomial $V(x,\delta)$ and if $N(\delta) \ge 0$ is a polynomial constraint, we can apply Lemma 3.4 and P-satz to (3.49) – (3.51) to get the following sufficient conditions:

Optimization Problem 3.5 (Parameter-dependent single Lyapunov function):

$$\max \beta \qquad \text{over } V \in \mathcal{R}_{n+n_\delta}, \quad s_1,\ldots,s_6 \in \Sigma_{n+n_\delta}$$

such that

$$V(x,\delta) - l_1(x) - s_1 N(\delta) \in \Sigma_{n+n_\delta} \tag{3.54}$$

$$(1 - V(x,\delta)) - s_2(\beta - p(x)) - s_3 N(\delta) \in \Sigma_{n+n_\delta} \tag{3.55}$$

$$-\frac{\partial V}{\partial x} f(x,\delta) s_6 - l_2(x) - s_4(1 - V) - s_5 N(\delta) \in \Sigma_{n+n_\delta} \tag{3.56}$$

Here, Σ_{n+n_δ} denotes the set of SOS polynomials in (x,δ). Since constraint (3.54) implies that for each δ such that $N(\delta) \ge 0$, $V(x,\delta)$ is underbounded by positive definite function $l_1(x)$, the set $\{x \in \mathbb{R}^n \,|\, V(x,\delta) \le 1\}$ is bounded for each δ.

3.3.2 Parameter-Dependent Composite Lyapunov Function

When the composite Lyapunov function $V(x,\delta) = \max_{i=1}^{q}\{V_i(x,\delta)\}$ is used, the following are sufficient conditions to find such a V:

Optimization Problem 3.6 (Parameter-dependent composite Lyapunov function):

$$\max \beta \qquad \text{over } V_i \in \mathcal{R}_{n+n_\delta}, \quad s_{0ij}, s_{1i}, \ldots, s_{6i} \in \Sigma_{n+n_\delta}$$

such that

$$V_i(x,\delta) - l_1(x) - s_{1i}N(\delta) \in \Sigma_{n+n_\delta} \tag{3.57}$$

$$(1 - V_i(x,\delta)) - s_{2i}(\beta - p(x)) - s_{3i}N(\delta) \in \Sigma_{n+n_\delta} \tag{3.58}$$

$$-\frac{\partial V_i}{\partial x} f(x,\delta) s_{6i} - l_2(x) - s_{4i}(1 - V_i) - s_{5i}N(\delta) - \sum_{\substack{j=1 \\ j\neq i}}^{q} s_{0ij}(V_i - V_j) \in \Sigma_{n+n_\delta} \tag{3.59}$$

There are $3q$ SOS constraints for this optimization problem.

Note that constraint (3.59) has the highest degree among (3.57) – (3.59). When $n+n_\delta$ and the degree of (3.59) are large, the large number of additional variables in the affine subspace can make optimizing (3.59) difficult. For example, when $n = 3$, $n_\delta = 1$, and degree $2d = 8$, $N_1 = 1990$ (see Table 2.1), which is considered a big computational problem for the present day computer.

We propose an ad-hoc two step process to reduce the number of decision variables for the case of a single uncertain parameter δ:

Step One:

Constraint (3.49) implies that $V(0,\delta) = 0$, so one possible parameterization for $V(x,\delta)$ is $z_1^T P_1 z_1 + \delta(z_2^T P_2 z_2) + \delta^2(z_3^T P_3 z_3) + \cdots$, where $z_i(x)$ are vectors of monomials of x. With this parameterization of $V(x,\delta)$, we will retain constraints (3.57) and (3.58) for the optimization problem, but modify (3.59) as follows: grid the set $\{\delta \,|\, N(\delta) \geq 0\}$ with m points and solve (3.59) for each value of $\bar{\delta}_k$. Using the same example, N_1 is now reduced to $126 \times m$ because δ does not enter as a variable, so there is a reduction in both n and $2d$. The gridded optimization problem below will have $2q$ SOS constraints in (x, δ) plus mq SOS constraints in x.

Optimization Problem 3.7 (Parameter-dependent gridded composite Lyapunov function):

$$\max \beta \qquad \text{over } V_i \in \mathcal{R}_{n+n_\delta},\ s_{1i}, s_{2i}, s_{3i} \in \Sigma_{n+n_\delta},\ s_{0ijk}, s_{4ik}, s_{6ik} \in \Sigma_n,\ k = 1, \ldots, m$$

such that

$$V_i(x,\delta) - l_1(x) - s_{1i} N(\delta) \in \Sigma_{n+n_\delta} \tag{3.60}$$

$$(1 - V_i(x,\delta)) - s_{2i}(\beta - p(x)) - s_{3i} N(\delta) \in \Sigma_{n+n_\delta} \tag{3.61}$$

$$-\frac{\partial V_i(x,\bar{\delta}_k)}{\partial x} f(x,\bar{\delta}_k) s_{6ik} - l_2(x) - s_{4ik}(1 - V_i(x,\bar{\delta}_k)) - \sum_{\substack{j=1 \\ j \neq i}}^{q} s_{0ijk}(V_i(x,\bar{\delta}_k) - V_j(x,\bar{\delta}_k)) \in \Sigma_n \tag{3.62}$$

Step Two:

After $V(x,\delta)$ is found and fixed, the second step is to solve (3.59) to obtain SOS multipliers $s_{0ij}, s_{4i}, s_{5i}, s_{6i} \in \Sigma_{n+n_\delta}$. With $V(x,\delta)$ fixed, (3.59) is longer bilinear in the decision polynomials and can be easily formulated as an SDP.

3.3.3 Parameter Independent Lyapunov Function

We can also search for δ independent polynomial Lyapunov function $V(x)$ for the uncertain system (3.48). We would expect that provable region of attraction $\{x \,|\, V(x) \leq 1\}$ would be smaller than provable region of attraction given by $V(x,\delta)$. However, a δ independent polynomial Lyapunov function allows for arbitrarily fast varying $\delta(t)$ whereas the parameter-dependent $V(x,\delta)$ does not.

The conditions for enlarging a provable region of attraction are:

Optimization Problem 3.8 (Parameter independent single Lyapunov function):

$$V(x) \text{ is positive definite,} \tag{3.63}$$

$$\text{the set } \{x \,|\, V(x) \le 1\} \text{ is bounded,}$$

$$\{x \,|\, p(x) \le \beta\} \subseteq \{x \,|\, V(x) \le 1\}, \text{ and} \tag{3.64}$$

$$\text{for each } \delta \text{ satisfying } N(\delta) \ge 0, \quad \{x \,|\, x \ne 0, V(x) \le 1\} \subseteq \{x \,|\, \tfrac{\partial V}{\partial x} f(x,\delta) < 0\}\,. \tag{3.65}$$

Again, applying P-satz to (3.63) – (3.65), we have the following sufficient conditions:

$$\max \beta \qquad \text{over } V \in \mathcal{R}_n, \quad s_2 \in \Sigma_n, \quad s_4, s_5, s_6 \in \Sigma_{n+n_\delta}$$

such that

$$V(x) - l_1(x) \in \Sigma_n \tag{3.66}$$

$$(1 - V(x)) - s_2(\beta - p(x)) \in \Sigma_n \tag{3.67}$$

$$-\tfrac{\partial V}{\partial x} f(x,\delta) s_6 - l_2(x) - s_4(1 - V) - s_5 N(\delta) \in \Sigma_{n+n_\delta} \tag{3.68}$$

When the composite Lyapunov function $V(x) = \max_{i=1}^{q}\{V_i\}$ is used, the following are sufficient conditions to find such a V:

Optimization Problem 3.9 (Parameter independent composite Lyapunov function):

$$\max \beta \qquad \text{over } V \in \mathcal{R}_n,\ V_i(0) = 0, \quad s_{2i} \in \Sigma_n, \quad s_{0ij}, s_{4i}, s_{5i}, s_{6i} \in \Sigma_{n+n_\delta}$$

such that

$$V_i(x) - l_1(x) \in \Sigma_n \tag{3.69}$$

$$(1 - V_i(x)) - s_{2i}(\beta - p(x)) \in \Sigma_n \tag{3.70}$$

$$-\tfrac{\partial V_i}{\partial x} f(x,\delta)s_{6i} - l_2(x) - s_{4i}(1 - V_i) - s_{5i}N(\delta) - \sum_{\substack{j=1 \\ j\neq i}}^{q} s_{0ij}(V_i - V_j) \in \Sigma_{n+n_\delta} \tag{3.71}$$

There are $3q$ SOS constraints for this optimization problem.

3.3.4 Examples

3.3.4.1 Example 1 - ROA of Uncertain 2-D Van der Pol

We revisit the 2-D Van der Pol equations, but with uncertainty in the $\dot{x}_1$ equation. Again, the same $p(x)$ as in (3.33) is used.

$$\dot{x}_1 = -(1 - 0.2\delta)x_2 \qquad\qquad \delta \in [-1, 1]$$

$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \tag{3.72}$$

Table 3.6 shows the β obtained for various fixed and known $\delta \in [-1, 1]$ using a single quartic $V(x)$ for optimization problem 3.1, and using pointwise maximum of two quartic $V(x)$'s for optimization problem 3.2. Since δ is known exactly, for a given degree of V, the smallest β value along a row in the table would be an upper bound for the achievable β using either parameter-dependent or independent Lyapunov functions for the uncertain system (3.72), where δ is not known.

We chose to search over a single $V(x, \delta)$ of the form $V(x, \delta) = z_1^T P_1 z_1 + \delta(z_2^T P_2 z_2)$, where $z_1^T P_1 z_1$ is a quartic polynomial in x only, and $z_2^T P_2 z_2$ is a quadratic polynomial

Table 3.6. Uncertain VDP: Measure of ROA for various δ

q	δ	-1.000	-0.667	-0.333	0.000	0.333	0.667	1.000
1	β	0.600	0.633	0.656	0.659	0.660	0.660	0.659
2	β	0.836	0.866	0.897	0.928	0.957	0.981	0.998

in x only. Using this form and optimization problem 3.6, we obtain $\beta = 0.600$, which is very close to the smallest β in Table 3.6. Figure 3.18 shows provable regions of attraction, $\{x \mid V(x, \delta) \leq 1\}$, for various values of δ and the corresponding limit cycles. Note that $\{x \mid p(x) \leq \beta = 0.600\}$ (the dashed ellipse) is contained in these provable ROAs.

For pointwise maximum of two quartic $V(x, \delta)$ of the same form, $\beta = 0.806$ is obtained, and Figure 3.19 shows these provable ROAs. Interestingly for some δ values, there are some points of the provable ROAs that are outside the limit cycles of other δ values. This demonstrates that provable ROAs from parameter-dependent Lyapunov functions are able to capture the different stability regions due to parameter variation.

The pointwise maximum result is obtained using the two step process described in Section 3.3.2. First, 7 evenly spaced grid points are chosen in $\delta \in [-1, 1]$ for gridded optimization problem 3.7, using V and SOS multipliers with degrees listed in the second row of Table 3.7, resulting in a 6th degree constraint (3.62) in x. After the pair of $V_i(x, \delta)$ is found, we verify that the original optimization problem 3.6 is feasible by attempting to find the SOS multipliers $s_{0ij}, s_{4i}, s_{5i}, s_{6i} \in \Sigma_{n+n_\delta}$ in constraint (3.59). A 6th degree constraint (3.59) returns infeasible results because the variations in $s_{0ijk}, s_{4ik}, s_{6ik}$ over δ are too large to be fitted with low degree SOS multipliers as listed in the third row of Table 3.7. An 8th degree constraint (3.59) in (x, δ) is needed to obtain feasible result (see row 4 of Table 3.7).

Using parameter independent $V(x)$, we obtain $\beta = 0.545$ for a single quartic V using optimization problem 3.8 and $\beta = 0.772$ for pointwise maximum of two quartic V's using optimization problem 3.9. These β values are smaller than those from $V(x, \delta)$, which is expected.

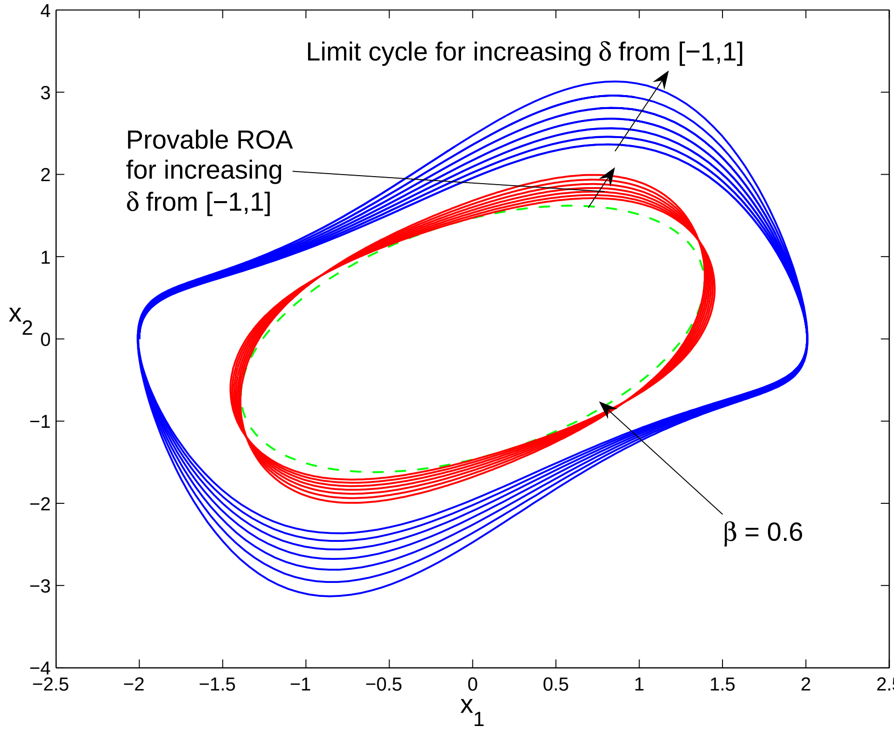

Figure 3.18. Provable ROAs using a single quartic $V(x, \delta)$

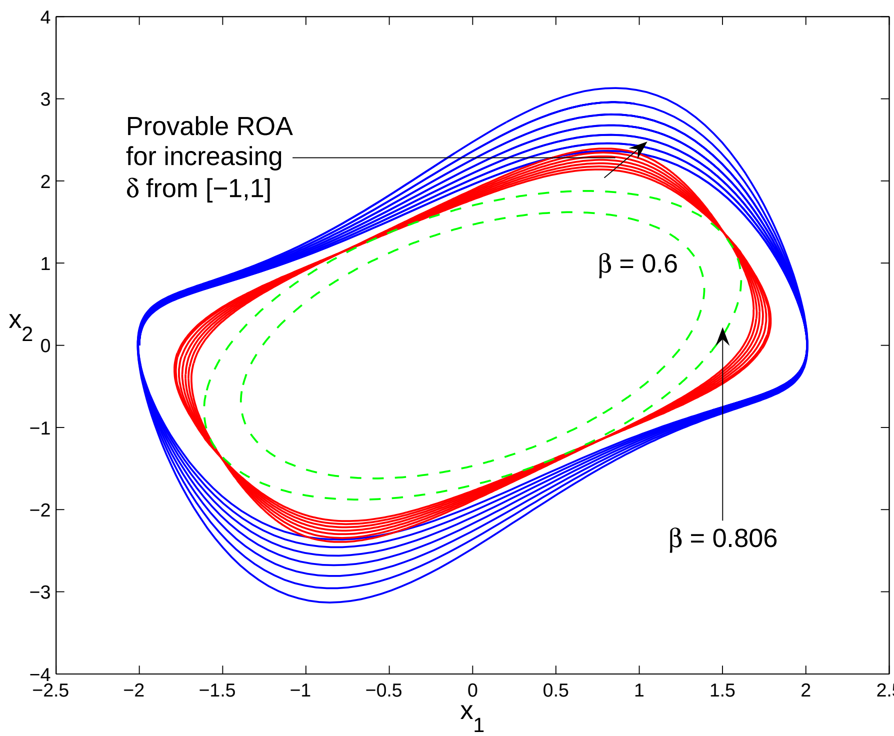

Figure 3.19. Provable ROAs using pointwise max of two quartic $V(x, \delta)$

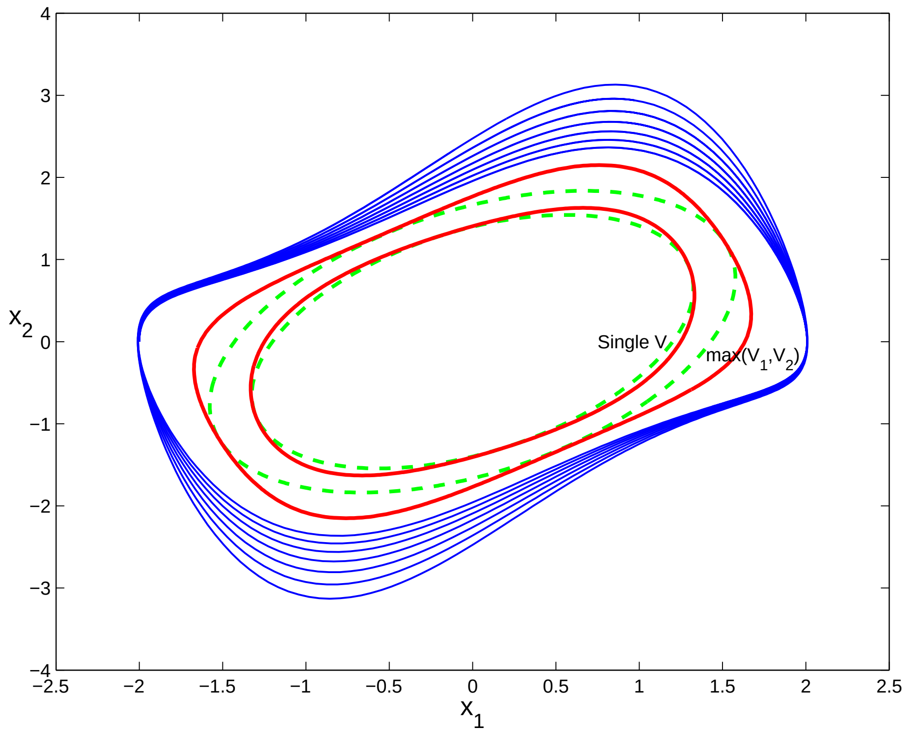

Figure 3.20. Provable ROAs using a single and pointwise max of $V(x)$

Table 3.7. Uncertain VDP: $V(x,\delta)$ and $V(x)$

Optimization Problem		q	degree of V	s_0	s_1	s_2	s_3	s_4	s_5	s_6	β	total no. of decision variables
$V(x,\delta)$	3.5	1	4	-	2	2	4	2	4	0	0.600	193
$V(x,\bar{\delta})$	3.7	2	4	2	2	2	4	2	-	0	0.806	607
$V(x,\delta)$	(3.59)	2	4	2	-	-	-	2	4	0	NaN	146
→ $V(x,\delta)$	(3.59)	2	4	4	-	-	-	4	6	2	0.806	473
$V(x)$	3.8	1	4	-	-	2	-	2	4	0	0.545	104
$V(x)$	3.9	2	4	2	-	2	-	2	4	0	0.772	223

3.3.4.2 Example 2 - ROA of Uncertain 3-D system

This example is (3.38) with uncertainties in the $\dot{x}_1$ and $\dot{x}_2$ equations. Again, the same $p(x)$ as in (3.39) is used.

$$\begin{aligned}\dot{x}_1 &= -x_2 - 0.45(1+\delta)x_3 \qquad \delta \in [-1,1] \\ \dot{x}_2 &= -x_3 - 0.45(1+\delta)x_2 \\ \dot{x}_3 &= -0.915x_1 + (1-0.915x_1^2)x_2 - x_3\end{aligned} \tag{3.73}$$

Table 3.8 shows the β values obtained with a single quartic V using optimization problem 3.1 and with pointwise maximum of two quartic V's using optimization problem 3.2, for known and fixed δ. To give the reader an idea of how much variation in the vector field for the uncertain system is, Figure 3.21 shows provable regions of attractions $\{x \,|\, V(x) \leq 1\}$ for various fixed values of δ on the $x_2 - x_3$ plane at $x_1 = 0$.

For parameter-dependent Lyapunov functions, we choose to search over quartic $V(x,\delta)$'s of the form $V(x,\delta) = z_1^T P_1 z_1 + \delta(z_2^T P_2 z_2)$, where $z_1^T P_1 z_1$ is a quartic polynomial in x only, and $z_2^T P_2 z_2$ is a quadratic polynomial in x only. A single quartic $V(x,\delta)$ yields $\beta = 2.93$ using optimization problem 3.6, while the pointwise maximum of two quartic $V(x,\delta)$ yields $\beta = 3.73$, which is quite close to the smallest value of $\beta = 4.15$ in Table 3.8.

Again, we use gridded optimization problem (3.7) for the pointwise maximum, with 7 evenly spaced grid points chosen in $\delta \in [-1,1]$ to solve for a 6th degree constraint (3.62) (see Table 3.9 for the degrees of V and SOS multipliers used). After finding the pair of $V(x,\delta)$, we need to verify that the original optimization problem 3.6 is feasible by attempting to

Table 3.8. Uncertain 3D example: Measure of ROA for various δ

q	δ	-1.000	-0.667	-0.333	0.000	0.333	0.667	1.000
1	β	6.32	11.60	13.55	13.41	11.60	8.41	4.16
2	β	6.40	11.84	13.76	13.57	11.61	8.50	4.15

Table 3.9. Uncertain 3D example: $V(x, \delta)$

Optimization Problem		q	degree of V	s_0	s_1	s_2	s_3	s_4	s_5	s_6	β	total no. of decision variables
$V(x, \delta)$	3.5	1	4	-	2	2	4	2	4	0	2.93	559
$V(x, \bar{\delta})$	3.7	2	4	2	2	2	4	2	-	0	3.73	2299
$V(x, \delta)$	(3.59)	2	4	2	-	-	-	2	4	0	NaN	350
$V(x, \delta)$	(3.59)	2	4	4	-	-	-	4	6	2	NaN	1353
$V(x, \delta)$	(3.59)	2	4	6	-	-	-	6	8	4	3.73	4879
$V(x)$	3.8	1	4	-	-	2	-	2	4	0	NaN	417
$V(x)$	3.9	2	4	2	-	2	-	2	4	0	NaN	857

find the SOS multipliers $s_{0ij}, s_{4i}, s_{5i}, s_{6i} \in \Sigma_{n+n_\delta}$ in constraint (3.59). We need to use a 10th degree constraint (3.59) in order to obtain feasibility, as lower degree constraints are not feasible. See rows 3 – 5 of Table 3.9 for the degrees of V and SOS multipliers used. Had we done a direct optimization of problem 3.6, constraint (3.59) would be 10th degree in 4 variables and we would have to deal with 7000 decision variables in the affine subspace of this constraint alone. Most likely, we would not have found a feasible solution.

No feasible solution was found for parameter independent quartic $V(x)$, which points to the usefulness of the parameter-dependent $V(x, \delta)$.

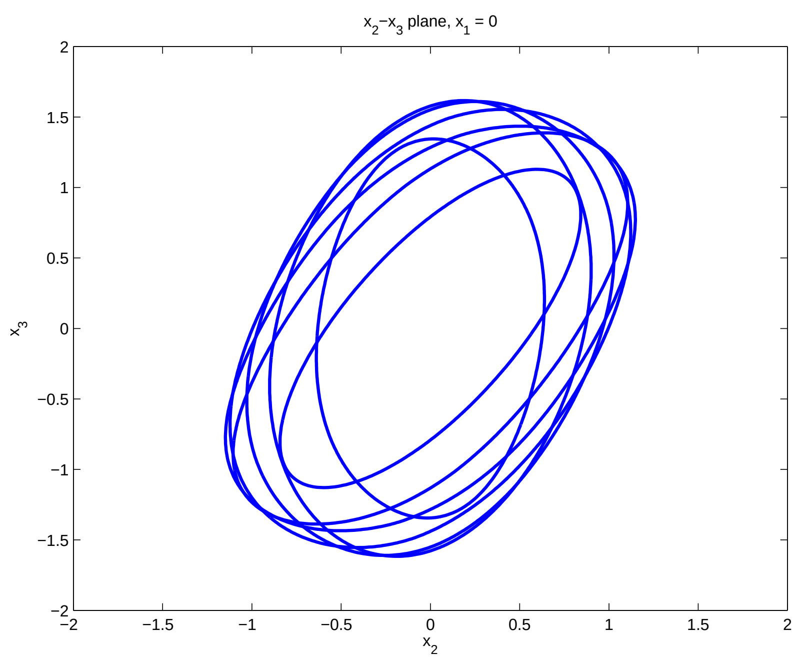

Figure 3.21. Provable ROAs using quartic $\max\{V_1(x), V_2(x)\}$ for fixed and known δ

3.4 Computation Statistics

In Section 3.1, we presented optimization problems that enlarge provable regions of attraction for polynomial systems and have given three examples to illustrate our methods. Unfortunately, there are several steps which make our approach a sufficient condition: searching for polynomial V's only, limiting the degrees of V and the SOS multipliers, and searching over non-convex decision variables space. Given these deficiencies, we want to investigate, in this section, how well the optimization problems and the bilinear solver perform in practice with respect to arbitrary data and increasing problem size. A benchmark example, inspired by Example 5 of [8], is chosen because the system has a known region of attraction and can be extended easily to any number of states. Let

$$\dot{x} = -Ix + (x^T Bx)x \tag{3.74}$$

where $x(t) \in \mathbb{R}^n$, and $B \in \mathbb{R}^{n\times n}$, $B \succ 0$.

This example has a special structure because the set $\{x \in \mathbb{R}^n \,|\, x^T Bx < 1\}$ is the exact region of attraction, which can be easily verified by noting that for each x_i, $\dot{x}_i = (-1+x^T Bx)x_i$. For all $x(t)$ such that $x(t)^T Bx(t) < 1$, each $\dot{x}_i$ differential equation is stable, so $x(t)$ decreases with time, while remaining inside the set $\{x \,|\, x^T Bx < 1\}$. Eventually, $x(t)$ will eventually decay to zero. Conversely, for any $x(t)$ such that $x(t)^T Bx(t) > 1$, each $\dot{x}_i$ is unstable, so $x(t)$ diverges away from the origin.

Let the set that we are interested in enlarging be $P_\beta := \{x \in \mathbb{R}^n \,|\, p(x) \le \beta\}$, where $p(x) := x^T Rx$ and $R \in \mathbb{R}^{n\times n}$, $R \succ 0$. For randomly selected positive definite B and R, we want to find how tight can we make P_β fit inside this region of attraction. For ease of analysis, we use the closure of this region of attraction, i.e. $\{x \in \mathbb{R}^n \,|\, x^T Bx \le 1\}$, so the containment condition is

$$\{x \in \mathbb{R}^n \,|\, x^T R x \leq \beta\} \subseteq \{x \in \mathbb{R}^n \,|\, x^T B x \leq 1\} \tag{3.75}$$

Applying $\mathcal{S}$-procedure to (3.75), the containment condition (3.75) holds iff there exists a $\lambda \geq 0$ such that

$$\begin{aligned} (1 - x^T B x) - \lambda(\beta - x^T R x) \geq 0 \\ \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} 1 - \lambda\beta & 0 \\ 0 & \lambda R - B \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0 \\ \Leftrightarrow \qquad 1 - \lambda\beta \geq 0 \qquad \text{and} \qquad \lambda R - B \succeq 0 \end{aligned}$$

The largest λ such that $1 - \lambda\beta \geq 0$ is $\lambda = \frac{1}{\beta}$. Substituting $\lambda = \frac{1}{\beta}$ into $\lambda R - B \succeq 0$, we have

$$\begin{aligned} \frac{1}{\beta} R - B \succeq 0 \\ \frac{1}{\beta} I - R^{-\frac{1}{2}} B R^{-\frac{1}{2}} \succeq 0 \\ \frac{1}{\beta} \geq \lambda_{\max}(R^{-\frac{1}{2}} B R^{-\frac{1}{2}}) \end{aligned}$$

So the best β that can be obtained from the optimization of this example is $\beta = [\lambda_{\max}(R^{-\frac{1}{2}} B R^{-\frac{1}{2}})]^{-1}$, or $\beta \times \lambda_{\max}(R^{-\frac{1}{2}} B R^{-\frac{1}{2}}) = 1$.

Since the exact region of attraction for this example is $\{x \in \mathbb{R}^n \,|\, x^T B x < 1\}$, a single quadratic Lyapunov function is all that is needed. We will use optimization problem 3.1 to search for the Lyapunov function. For each n, the size of the problem, we perform 100 trials where random, positive definite B and R were picked. Each B and R has eigenvalues $\exp(2r_i)$ where each r_i is picked from a normal distribution with zero mean and unit variance. For each trial, we run the optimization 3 times, so for each n, there is a total of 300 runs. Table 3.10 shows the results of the test.

Table 3.10. Computation statistics for the benchmark example

n	Number of variables	Number of success	Number of failures	Excluding unsuccessful runs, worst case $\beta \times \lambda_{\max}$ over 300 runs	Excluding unsuccessful runs, worst case $\beta \times \lambda_{\max}$ over 100 trials	Average time per run (secs)
2	13	298	2	0.99995	1.00000	0.70
3	25	296	4	0.90955	0.99984	1.12
4	48	297	3	0.07687	0.99999	2.14
6	157	297	3	0.99997	0.99998	11.18
8	420	300	0	0.99989	0.99992	99.69

A run is considered successful if the solver returns the message "No problems detected", even though the results might not be the global optimum. A run is classified as a failure if the solver either returns "Infeasible", "Numerical problems" or "Maximum number of iterations exceeded". The latter means the solver fails to converge to the required accuracy after the maximum number of iterations is reached (in this case, 250 iterations). Except for the case of $n = 6$, there are no trials that fail for all 3 runs. For the case of $n = 6$, one trial failed in all 3 runs because that particular randomly generated P and B lead to poor numerical conditioning in the optimization problem. A more relaxed bound on the entries of the decision variables is used to circumvent this problem. When those failed runs were re-run, we did not encounter any problems and we obtained β values such that $\beta \times \lambda_{\max}(R^{-\frac{1}{2}} B R^{-\frac{1}{2}}) \approx 1$.

Among the successful runs, we are interested in how close the results are to the optimal value. Under column "worst case $\beta \times \lambda_{\max}$ over 300 runs" we have the worst case $\beta \times \lambda_{\max}(R^{-\frac{1}{2}} B R^{-\frac{1}{2}})$ value among the 300 runs that are successful. The next column shows the worst case $\beta \times \lambda_{\max}$ over 100 trials, which means for each trial, take the maximum β over 3 runs, and present the worst trial result among the 100 trials. Since the results of this column are ≈ 1, this indicates that repeated runs of the same problem eventually lead to the optimal solution for this example. The reason why repeated runs lead to different results is because PENBMI starts the bilinear optimization with randomized initial conditions, so if the number of decision variables is small, the chances of an initial condition starting in the region where the solution will converge to the global optimum is high.

For this benchmark example, we can see that even though our problem formulation is bilinear in the decision polynomials and the bilinear solver is a local solver, the results obtained are encouraging.

3.5 Chapter Summary

In this chapter, we presented the techniques of using sum-of-squares programming for finding provable regions of attraction and attractive invariant sets for nonlinear systems with polynomial vector fields. The examples presented yield results that are, practically speaking, not conservative. Moreover, the composite Lyapunov function method has the advantage of reducing the number of decision variables. We hope that such reductions will enable us to apply these techniques to higher degree systems in more variables before the curse of dimensionality renders the method impractical.

We also formulated the techniques of enlarging a provable region of attraction for polynomial systems with uncertainty using both parameter-dependent and independent Lyapunov functions. Besides the use of composite Lyapunov functions, an ad-hoc two-step optimization process is proposed to further reduce the number of decision variables. The first step is to solve an optimization problem in the gridded uncertain parameter space to obtain the Lyapunov functions. This is then followed by verification of the original, ungridded problem by computing the SOS multipliers.

Finally, we presented computation statistics of a benchmark example of enlarging its provable region of attraction with arbitrary data and increasing problem size.

Chapter 4

Performance Analysis

In this chapter, we study the local input-to-output gain of nonlinear systems using sum-of-squares optimization. First, we derive an upper bound of the reachable set due to a bounded $\mathcal{L}_2$ disturbance and present improvement to the upper bound using sum-of-squares techniques. An example used in our previous work is revisited to show the improvement in the upper bound and the tightness of this bound when compared to the lower bound obtained by a power-like algorithm. We also consider the problem of finding an upper bound of the reachable set due to an $\mathcal{L}_2$ disturbance with an $\mathcal{L}_\infty$ bound. This problem is of practical interest as in most situations, a disturbance has finite amplitude and energy.

The second part of this chapter derives an upper bound of the induced $\mathcal{L}_2 \to \mathcal{L}_2$ gain for a nonlinear system. We also propose a refinement of this induced gain by using pointwise maximum of polynomials. Finally, we present an interesting example analyzing the induced gain of an adaptive control system by Krstić [21] and the effects of adaptation gain on this induced gain.

4.1 Reachable Set with $\mathcal{L}_2$ Disturbances

A SOS programming-based algorithm for finding an upper bound for the reachable set of a nonlinear system under bounded $\mathcal{L}_2$ disturbances was presented in [16], using a variant of the barrier function approach introduced in [41]. Here, we show how the obtained upper bound can usually be further improved using semidefinite programming.

Given a system of the form

$$\dot{x} = f(x, w) \tag{4.1}$$

with $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, and f is a n-vector with elements of $\mathcal{R}_{n+n_w}$ such that $f(0,0) = 0$. Compute a bound on the set of points $x(T)$ that are reachable from $x(0) = 0$ under (4.1), provided the disturbance satisfies $\int_0^T w(t)^T w(t)\, dt \leq R^2$, $T \geq 0$. Following the Lyapunov-like argument in [5, §6.1.1], if a continuously differentiable function V satisfies

$$V(x) > 0 \text{ for all } x \in \mathbb{R}^n \backslash \{0\} \text{ with } V(0) = 0, \text{ and} \tag{4.2}$$

$$\tfrac{\partial V}{\partial x} f(x, w) \leq w^T w \text{ for all } x \in \mathbb{R}^n, w \in \mathbb{R}^{n_w}, \tag{4.3}$$

then $\{x \,|\, V(x) \leq R^2\}$ contains the set of points $x(T)$ that are reachable from $x(0) = 0$ for any w such that $\int_0^T w(t)^T w(t)\, dt \leq R^2$, $T \geq 0$. We can see this by integrating the inequality in (4.3) from 0 to T, yielding

$$V(x(T)) - V(x(0)) \leq \int_0^T w(t)^T w(t)\, dt \leq R^2. \tag{4.4}$$

Recalling $V(x(0)) = 0$, $x(T) \in \{x \,|\, V(x) \leq R^2\}$. Furthermore, $x(\tau) \in \{x \,|\, V(x) \leq R^2\}$ for all $\tau \in [0, T]$, allowing us to relax the inequality in (4.3) to

$$\tfrac{\partial V}{\partial x} f(x, w) \leq w^T w \quad \forall x \in \{x \,|\, V(x) \leq R^2\}, \forall w \in \mathbb{R}^{n_w}. \tag{4.5}$$

A more precise proof of this relaxation is given in Lemma 4.1 of Section 4.1.1.

Since we're interested in a tight upper bound for the reachable set, we introduce a variable sized region $P_\beta := \{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\}$ and require $\{x \,|\, V(x) \leq R^2\} \subseteq P_\beta$ while minimizing β. The positive definite function p is chosen by the user to reflect the relative importance of the states.

If $l_1(x)$ is a given positive definite polynomial, then the constraint $x \neq 0$ in (4.2) is equivalent to $l_1(x) \neq 0$. Usually, we take $l_1(x)$ of the form $l_1(x) = \sum_{j=1}^n \epsilon_j x_j^2$, where ϵ_j are positive numbers. Applying the generalized $\mathcal{S}$-procedure (Lemma 2.1) to each of the constraints (4.2), (4.5), and the set containment condition, yields the bilinear SOS optimization below:

Optimization Problem 4.1 (Reachable set under bounded $\mathcal{L}_2$ disturbances):

$$\min \beta \quad \text{over } V \in \mathcal{R}_n,\ V(0) = 0,\ s_4 \in \Sigma_n,\ s_{10} \in \Sigma_{n+n_w}$$

such that

$$V - l_1 \in \Sigma_n, \tag{4.6}$$

$$(\beta - p) - (R^2 - V)s_4 \in \Sigma_n, \tag{4.7}$$

$$-\left((R^2 - V)s_{10} + \tfrac{\partial V}{\partial x} f(x,w) - w^T w\right) \in \Sigma_{n+n_w}. \tag{4.8}$$

4.1.1 Upper Bound Refinement

Optimization problem 4.1 constrains the reachability of x under $\mathcal{L}_2$ disturbances, but the upper bound is a crude one: in (4.5), we require that $\frac{\partial V}{\partial x} f(x,w) \leq w^T w$ for all x in the entire region of $\{x \,|\, V(x) \leq R^2\}$, but if we subdivide $\{x \,|\, V(x) \leq R^2\}$ into smaller annular regions, we might be able to refine this upper bound. The next lemma proposes this refinement.

Lemma 4.1. *Suppose that for* $k = 1, 2, \ldots, m$*,*

$$\tfrac{\partial V}{\partial x} f \leq h_k w^T w \ \ \textit{on} \ \{x \in \mathbb{R}^n \,|\, (k-1)\epsilon \leq V(x) \leq k\epsilon\} \ \ \textit{and for all } w \in \mathbb{R}^{n_w}. \tag{4.9}$$

Then for any $k \leq m$*, the dynamic system* $\dot{x} = f(x, w)$ *has the property: Starting from* $x(0) = 0$*, for piecewise continuous* w*,*

$$\int_0^T w^T w \, dt < \epsilon \left(\frac{1}{h_1} + \cdots + \frac{1}{h_k} \right) \Rightarrow V(x(T)) \leq k\epsilon.$$

Proof. Suppose not, i.e. $\exists T > 0$ such that $V(T) > k\epsilon$. By continuity of V, and $V(0) = 0$, $V(T) > k\epsilon$, there exist t_j and $\hat{t}_j$, with $0 < \hat{t}_1 \leq t_1 < \hat{t}_2 \leq t_2 < \ldots \leq t_{k-1} < \hat{t}_k < T$ such that for all $j = 1, 2, \ldots, k$,

$$V(\hat{t}_j) = j\epsilon, \ V(t_{j-1}) = (j-1)\epsilon, \text{ and } \forall t \in [t_{j-1}, \hat{t}_j], \ (j-1)\epsilon \leq V(t) \leq j\epsilon. \tag{4.10}$$

Now,

$$\begin{aligned}
\int_0^T w^T w \, dt &= \int_0^{\hat{t}_1} w^T w \, dt + \int_{\hat{t}_1}^{t_1} w^T w \, dt + \int_{t_1}^{\hat{t}_2} w^T w \, dt + \cdots + \int_{t_{k-1}}^{\hat{t}_k} w^T w \, dt + \int_{\hat{t}_k}^{T} w^T w \, dt \\
&\geq \int_0^{\hat{t}_1} w^T w \, dt + \int_{t_1}^{\hat{t}_2} w^T w \, dt + \cdots + \int_{t_{k-1}}^{\hat{t}_k} w^T w \, dt \\
&\geq \frac{1}{h_1} \int_0^{\hat{t}_1} \dot{V} \, dt + \frac{1}{h_2} \int_{t_1}^{\hat{t}_2} \dot{V} \, dt + \cdots + \frac{1}{h_k} \int_{t_{k-1}}^{\hat{t}_k} \dot{V} \, dt \qquad \text{by (4.9)} \\
&\geq \epsilon \sum_{j=1}^{k} \frac{1}{h_j} \qquad\qquad \text{because} \int_{t_{j-1}}^{\hat{t}_j} \dot{V} \, dt = \epsilon \qquad \text{by (4.10)}
\end{aligned}$$

So $V(x(T)) > k\epsilon \Rightarrow \int_0^T w^T w \, dt \geq \epsilon \sum_{j=1}^{k} \frac{1}{h_j}$. □

The upper bound refinement is as follows: given V obtained from optimization problem 4.1, which satisfies (4.6) – (4.8), for each $k = 1, \ldots, m$, with $\epsilon m = R^2$, solve

Optimization Problem 4.2 (Reachable set refinement):

$$\min\ h_k \quad \text{over} \quad s_{11k}, s_{12k} \in \Sigma_{n+n_w}$$

such that

$$-\left[(\epsilon k - V)s_{11k} + (V - \epsilon(k-1))s_{12k} + \tfrac{\partial V}{\partial x} f(x,w) - h_k w^T w\right] \in \Sigma_{n+n_w} \tag{4.11}$$

to conclude that

$$\int_0^T w^T w\, dt < \epsilon \sum_{k=1}^{m} \frac{1}{h_k} \Rightarrow p(x(t)) \le \beta \quad \forall t \le T.$$

With V fixed from the initial optimization problem 4.1, the refinement optimization problem 4.2 is simply an SDP, as the decision variables in the polynomials s_{11k} and s_{12k} enter linearly. By applying Lemma 2.1, constraint (4.11) is sufficient for (4.9).

4.1.2 Lower bound

For any positive T, it follows that

$$\max_{\substack{w \in \mathcal{L}_2[0,T] \\ \|w\|_2 \le R}} p(x(T)) \le \max_{\substack{w \in \mathcal{L}_2[0,\infty) \\ \|w\|_2 \le R}} p(x(t)) \le \beta \tag{4.12}$$

where β is an upper bound obtained from Section 4.1.

The conditions for stationarity of the finite horizon maximum in (4.12) are the existence of signals (x, λ) and w which satisfy $\dot{x} = f(x,w)$, $\|w\|_2 = R$, and

$$\begin{aligned}
\lambda(T) &= \left.\frac{\partial p}{\partial x}\right|_{x(T)}^{T} \\
\dot{\lambda}(t) &= -\left.\left(\frac{\partial f}{\partial x}\right)\right|_{\substack{x(t)\\ w(t)}}^{T} \lambda(t), \quad t \in [0,T] \\
w(t) &= \mu \left.\left(\frac{\partial f}{\partial w}\right)\right|_{\substack{x(t)\\ w(t)}}^{T} \lambda(t), \quad t \in [0,T]\ .
\end{aligned}$$

Tierno et al. [35], propose a power-like method to solve a similar maximization. Adapting the method for this case yields the following algorithm:

1. Pick w, with $\|w\|_2 = R$.

2. Compute solution of $\dot{x} = f(x, w)$, from $x(0) = 0$.

3. Set $\lambda(T) := \left.\frac{\partial p}{\partial x}\right|_{x(T)}^{T}$.

4. Compute solution of $\dot{\lambda}(t) = -\left(\frac{\partial f}{\partial x}\right)\Big|_{\substack{x(t)\\w(t)}}^{T} \lambda(t), \qquad t \in [0, T]$.

5. Update $w(t) = \mu \left(\frac{\partial f}{\partial w}\right)\Big|_{\substack{x(t)\\w(t)}}^{T} \lambda(t)$, with μ chosen so that $\|w\|_2 = R$.

6. Repeat steps 2 – 5 until w converges.

In practice, step 2 of each iteration gives a valid lower bound on maximum (over $\|w\|_2 = R$) of $p(x(T))$, independent of whether the iteration converges. A main point of [35] is that if the dynamics (i.e., f) are linear, and the functional p quadratic, then the iteration proposed is indeed the correct power iteration to compute the induced norm of the operator mapping w to $p(x(T))$.

4.1.3 Example

We use the example in [16] and [15] for comparison purposes. The nonlinear system is

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - x_1 x_2^2 \\ \dot{x}_2 &= -x_2 - x_1^2 x_2 + w \end{aligned} \tag{4.13}$$

with $x(t) \in \mathbb{R}^2$ and $w(t) \in \mathbb{R}$. We choose $p(x) = 8x_1^2 - 8x_1x_2 + 4x_2^2$. For the upper bound refinement, we use $\epsilon = 0.25$ (i.e. for each R^2, there are $4R^2$ annuli).

The original upper bound is computed at 13 different values of R^2, using optimization problem 4.1 and searching over quadratic V, quadratic (and homogeneous) s_{10} and constant (zero degree) s_4. The refined upper bound uses (at each value of R^2) the obtained V, and solves the SDP optimization problem 4.2 with quadratic s_{11k} and s_{12k}.

Figure 4.1 shows the original upper bound in [15], the new refined upper bound, as well as the lower bound[1] for the nonlinear system. Clearly, for $R^2 \leq 6$, the upper and lower bounds are quite tight, indicating that the upper bound refinement is very effective in this example. Moreover, the improvement due to the refinement is very clear for $R^2 > 4$. Also shown in Figure 4.1 is the exact answer for the linearization (about $x = 0, w = 0$) of (4.13). This is easily computed with grammians (see [5, Section 6.1, pg 78]). More interestingly, if the worst-case input for this linearized system is applied to system (4.13), the resultant peak value of $p(x(t))$ is quite suboptimal. This data is plotted, and is the lowest curve on the graph. In many cases, it is about 45% lower than the actual worst-case cost. These observations demonstrate that using linearized analysis for nonlinear systems with large disturbances can lead to inaccurate estimates.

Figure 4.2 shows a particular worst case $w(t)$ and the corresponding $p(x(t))$ for $\|w\|_2^2 \leq 6$ on the interval $t \in [0, 10]$.

4.1.4 Incorporating $\mathcal{L}_\infty$ constraints on w

Suppose for the reachable set problem, we impose an additional $\mathcal{L}_\infty$ constraint on w, say $w^T(t)w(t) \leq \gamma$ for all $t \geq 0$. Constraint (4.5) would be modified as follows:

$$\frac{\partial V}{\partial x} f(x, w) \leq w^T w \quad \forall x \in \{x \mid V(x) \leq R^2\}, \ \forall w \in \{w \mid w^T w \leq \gamma\}. \tag{4.14}$$

[1] Special thanks to Tim Wheeler for the lower bound calculations.

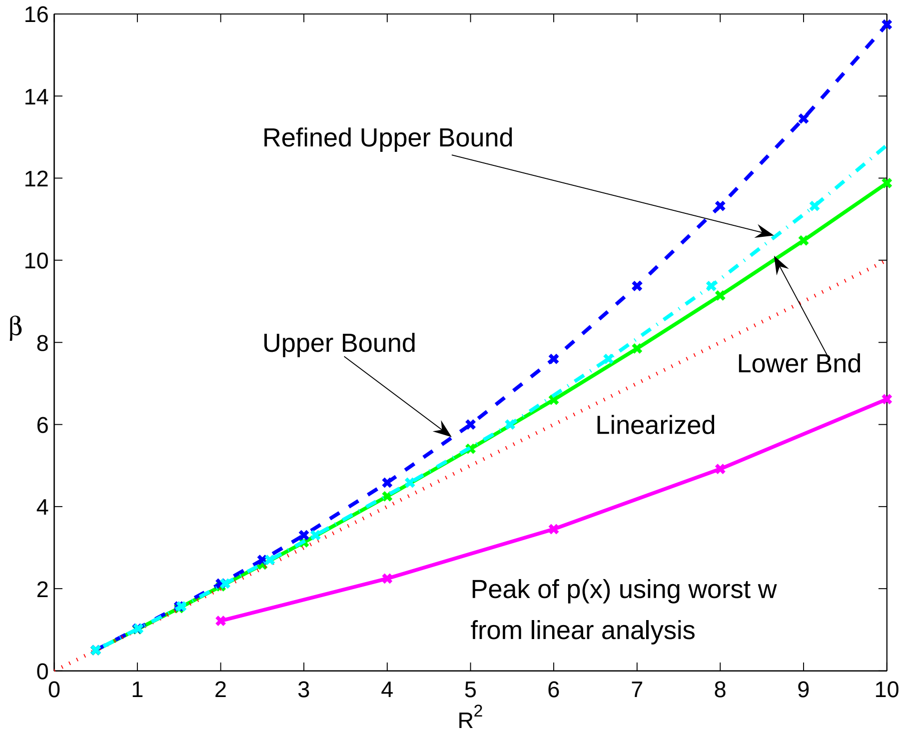

Figure 4.1. Bounds on reachable sets for various $\|w\|_2^2$

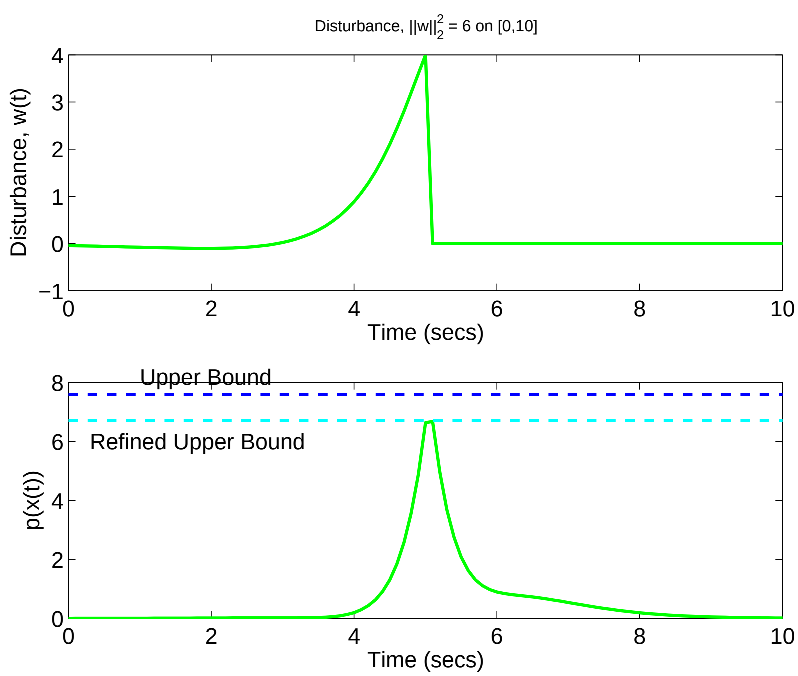

Figure 4.2. For $\|w\|_2^2 = 6$, top: $w(t)$, bottom: $p(x(t))$

Applying Lemma 2.1 to (4.14), we have the following optimization problem: For a fixed $\gamma > 0$,

Optimization Problem 4.3 (Reachable set under bounded $\mathcal{L}_2$ and $\mathcal{L}_\infty$ disturbances):

$$\min \beta \quad \text{over } V \in \mathcal{R}_n,\ V(0) = 0,\ s_4 \in \Sigma_n,\ s_{10}, s_{13} \in \Sigma_{n+n_w}$$

such that

$$V - l_1 \in \Sigma_n, \tag{4.15}$$

$$(\beta - p) - (R^2 - V)s_4 \in \Sigma_n, \tag{4.16}$$

$$-\left((R^2 - V)s_{10} + \frac{\partial V}{\partial x} f(x,w) - w^T w\right) - s_{13}(\gamma - w^T w) \in \Sigma_{n+n_w}. \tag{4.17}$$

The corresponding upper bound refinement is as follows: given $\gamma > 0$ and V satisfying (4.15) – (4.17), for each $k = 1, \ldots, m$, with $\epsilon m = R^2$, solve

Optimization Problem 4.4 (Reachable set refinement under bounded $\mathcal{L}_2$ and $\mathcal{L}_\infty$ disturbances):

$$\min h_k \quad \text{over} \quad s_{11k}, s_{12k}, s_{13k} \in \Sigma_{n+n_w}$$

such that

$$\begin{aligned} &-\left[(\epsilon k - V)s_{11k} + (V - \epsilon(k-1))s_{12k} + \frac{\partial V}{\partial x} f(x,w) - h_k w^T w\right] \\ &\quad - s_{13k}(\gamma - w^T w) \in \Sigma_{n+n_w}. \end{aligned} \tag{4.18}$$

We reuse the example in Section 4.1.3 to illustrate the effects of incorporating $\mathcal{L}_\infty$ constraints on w. We choose to search over quadratic V, quadratic (and homogeneous) s_{10} and s_{13}, and constant (zero degree) s_4. After constraining, for example $\|w\|_\infty \leq 2$, both the upper bound and the refined upper bound are much lower (see Figure 4.3), illustrating that as sufficient conditions, optimization problems 4.3 and 4.4 successfully exploit the $\mathcal{L}_\infty$

bound on w. In fact, for $R^2 > 7.5$, the refined upper bound is even lower than the bound for the linearized system. For clarity, we only show $R^2 \in [4, 10]$ as the curves for $R^2 < 4$ are close together.

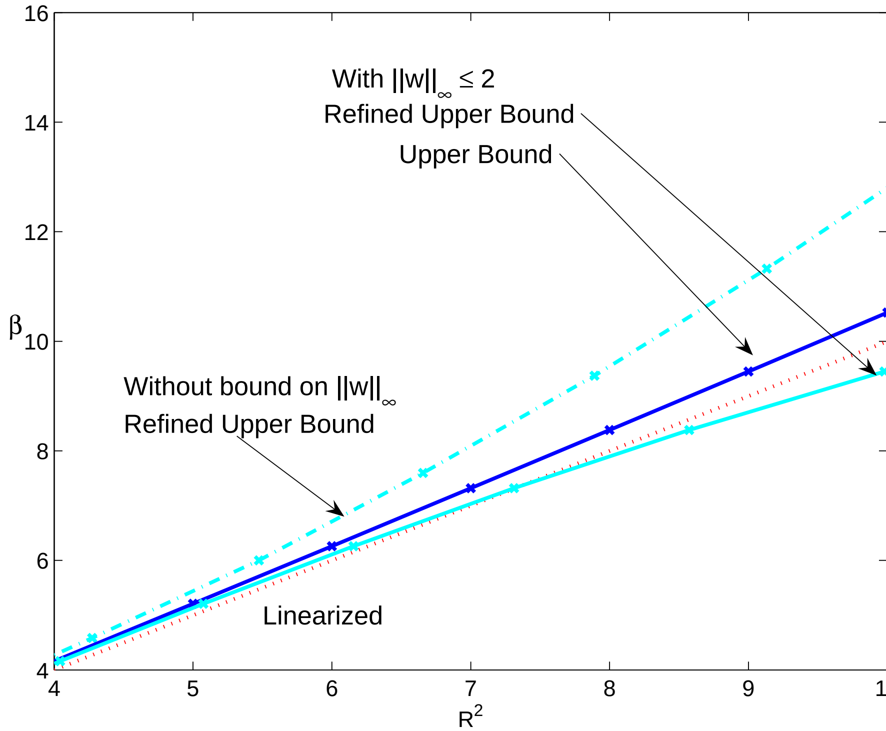

Figure 4.3. Bounds on reachable sets with and without bounds on $\|w\|_\infty$

4.2 $\mathcal{L}_2$ to $\mathcal{L}_2$ analysis of nonlinear systems

Consider the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x, w) \\ z &= h(x) \end{aligned} \tag{4.19}$$

with $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, and $z(t) \in \mathbb{R}^{n_z}$. f is an n-vector with elements of $\mathcal{R}_{n+n_w}$ such that $f(0, 0) = 0$, while h is an n_z-vector with elements of $\mathcal{R}_n$ such that $h(0) = 0$. We want to find upper and lower bounds for the w to z induced $\mathcal{L}_2$ to $\mathcal{L}_2$ gain of this system.

4.2.1 Upper Bound

Lemma 4.2. *Consider system* (4.19)*. If there exists a* $\gamma > 0$ *and a continuously differentiable function* V *such that*

$$V(0) = 0 \text{ and } V(x) \geq 0, \tag{4.20}$$

$$\frac{\partial V}{\partial x} f(x,w) \leq w^T w - \gamma^{-2} z^T z \quad \forall (x,w) \in \mathbb{R}^n \times \mathbb{R}^{n_w} \tag{4.21}$$

then for $x(0) = 0$ *and* $\|w\|_2 \leq R$*, we have* $\|z\|_2 \leq \gamma R$*.*

Proof. Integrate (4.21) from 0 to T,

$$\int_0^T \frac{\partial V}{\partial x} f(x,w)\, dt \leq \int_0^T (w^T w - \gamma^{-2} z^T z)\, dt$$
$$V(x(T)) - V(x(0)) \leq \int_0^T (w^T w - \gamma^{-2} z^T z)\, dt.$$

Since $V(x(T)) \geq 0$, and $V(x(0)) = V(0) = 0$,

$$\int_0^T \gamma^{-2} z^T z\, dt \leq \int_0^T w^T w\, dt \leq R^2$$
$$\Rightarrow \|z\|_2 \leq \gamma R \quad \text{or} \quad \frac{\|z\|_2}{\|w\|_2} \leq \gamma$$

□

Also note that $V(x(T)) \leq \int_0^T (w^T w - \gamma^{-2} z^T z) dt \leq \int_0^T w^T w dt \leq R^2$, therefore $V(x(T)) \leq R^2 \ \forall T \geq 0$, so we can relax (4.21) to hold for all $x \in \{x \,|\, V(x) \leq R^2\}$ instead of for all $x \in \mathbb{R}^n$. With this relaxation, Lemma 4.2 can be cast as the following optimization problem:

$$\min \gamma \qquad \text{over } V \in \Sigma_n,\ V(0) = 0$$

such that

$$\frac{\partial V}{\partial x} f(x,w) \leq w^T w - \gamma^{-2} z^T z \qquad \forall x \in \{x \,|\, V(x) \leq R^2\} \text{ and } \forall w \in \mathbb{R}^{n_w} \tag{4.22}$$

Applying generalized $\mathcal{S}$-procedure (Lemma 2.1) to (4.22), we have

Optimization Problem 4.5 (Upper bound for $\mathcal{L}_2$ to $\mathcal{L}_2$ gain):

$$\min \gamma \qquad \text{over } V \in \Sigma_n,\ V(0)=0, s_1 \in \Sigma_{n+n_w}$$

such that

$$-\left[(R^2-V)s_1 + \tfrac{\partial V}{\partial x} f(x,w) - w^T w + \gamma^{-2} z^T z\right] \in \Sigma_{n+n_w} \tag{4.23}$$

We can obtain a tighter upper bound if we use composite $V = \max_{i=1}^{q}\{V_i\}$. The conditions are derived following along the same line as in Section 3.1.2:

$$\min \gamma \qquad \text{over } V_i \in \Sigma_n,\ V_i(0)=0,\ i=1,\ldots,q$$

such that when $V_i(x) \geq V_j(x),\ j \neq i$,

$$\tfrac{\partial V_i}{\partial x} f(x,w) \leq w^T w - \gamma^{-2} z^T z \qquad \forall x \in \{x \,|\, V_i(x) \leq R^2\} \quad \text{and } \forall w \in \mathbb{R}^{n_w} \tag{4.24}$$

Applying Lemma 2.1 to (4.24), we have

Optimization Problem 4.6 (Upper bound for $\mathcal{L}_2$ to $\mathcal{L}_2$ gain using pointwise max of V):

$$\min \gamma \qquad \text{over } V_i \in \Sigma_n,\ V_i(0)=0,\ s_{0ij},\, s_{1i} \in \Sigma_{n+n_w}, \quad i=1,\ldots,q$$

such that

$$-\left[(R^2-V_i)s_{1i} + \tfrac{\partial V_i}{\partial x} f(x,w) - w^T w + \gamma^{-2} z^T z\right] - \sum_{\substack{j=1\\ j\neq i}}^{q} s_{0ij}(V_i - V_j) \in \Sigma_{n+n_w} \tag{4.25}$$

There are q SOS constraints for this optimization problem.

4.2.2 Lower Bound

For any positive T, it follows that

$$\max_{\substack{w\in\mathcal{L}_2[0,T]\\ \|w\|_2\leq R}} \|z\|_{2,[0,T]} \leq \max_{\substack{w\in\mathcal{L}_2[0,\infty)\\ \|w\|_2\leq R}} \|z\|_2 \leq \gamma \tag{4.26}$$

where γ is an upper bound obtained from (4.23). A power method to find stationary points of the finite horizon maximization is introduced in [35] and is similar to the algorithm presented in Section 4.1.2.

4.2.3 Example: $\mathcal{L}_2$ to $\mathcal{L}_2$ Analysis for an Adaptive Control System

Consider the simplest form of the problem studied in [21]:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta\phi(x_1) \\ \dot{x}_2 &= u \end{aligned} \tag{4.27}$$

where θ is an unknown constant parameter (set to 2 for this particular example) and $\phi := x_1 + 1$. The control objective is to regulate x_1 to $x_1^e := 0$.

The adaptive controller design procedure as documented in [21] is as follows (with design parameters c_1 and c_2 set to 1): let $z_1 := x_1$, $z_2 := x_2 - \alpha_1$, and $\alpha_1 := -z_1 - \hat{\theta}\phi(z_1)$. The controller state $\hat{\theta}$ evolves as

$$\dot{\hat{\theta}} = \tau_2 = \Gamma\left(z_1\phi - z_2\tfrac{\partial\alpha_1}{\partial z_1}\phi\right) ,$$

producing output u:

$$u = -z_2 - z_1 + \tfrac{\partial\alpha_1}{\partial z_1}(z_2 + \alpha_1 + \hat{\theta}\phi(z_1)) + \tfrac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2 .$$

From [21], for all θ, the closed loop system has a single globally asymptotically stable equilibrium point: $x_1^e = 0$, $x_2^e = -\theta\phi(x_1^e) = -\theta$ and $\hat{\theta}^e = \theta$.

Consider an input disturbance w, as shown in Figure 4.4. Starting from the equilibrium point $[z_1, z_2, \theta_e]^T = [0, 0, 0]^T$, what is the $\mathcal{L}_2 \to \mathcal{L}_2$ gain from w to x_1, and how is this affected by the adaptation gain Γ?

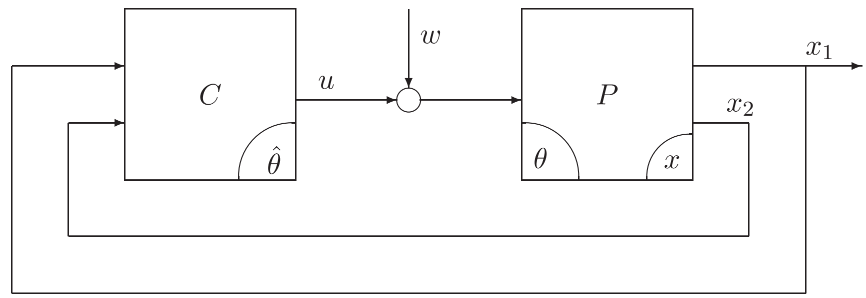

Figure 4.4. Block diagram for adaptive control example

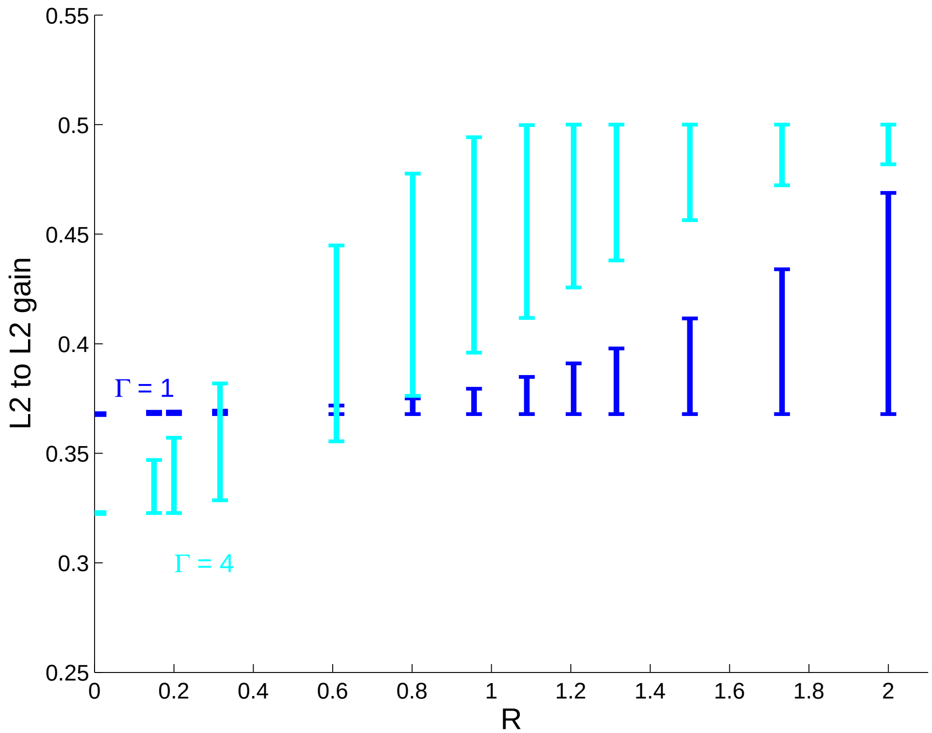

Figure 4.5. Krstić example: $\mathcal{L}_2$ to $\mathcal{L}_2$ gain $(w \to x_1)$ for $\Gamma = 1$ and $\Gamma = 4$

Figure 4.5 shows upper and lower bounds[2] on the induced $\mathcal{L}_2 \to \mathcal{L}_2$ gain for $\Gamma = 1$ and $\Gamma = 4$ at many different values of $\|w\|_2$. The upper bounds are obtained by using the pointwise maximum of two quartic V's for optimization problem 4.6.

The points marked at $\|w\|_2 = 0$ are the $\mathcal{H}_\infty$ norm of the linearized system. For positive values of $\|w\|_2$, upper and lower bounds on the induced gain are computed, and shown as interval bounds. This figure indicates that when the input disturbance is small ($\|w\|_2 \leq 0.22$,

[2]Special thanks to Tim Wheeler for the lower bound calculations.

Table 4.1. Krstić example: $\mathcal{L}_2$ to $\mathcal{L}_2$ gains (γ) of single and pointwise max of 2 quartic V's.

Γ	q	R	0.802	0.956	1.09	1.21	1.315	1.50	1.73	2.00
1	1	γ	0.380	0.386	0.393	0.400	0.408	0.424	0.449	0.486
	2	γ	0.375	0.379	0.385	0.391	0.398	0.412	0.434	0.469
4	1	γ	0.482	0.496	0.500	0.500	0.500	0.500	0.500	0.500
	2	γ	0.478	0.494	0.500	0.500	0.500	0.500	0.500	0.500

say), large adaptation gain ($\Gamma = 4$) has better worst-case disturbance rejection performance than small adaptation gain ($\Gamma = 1$). However, for large disturbances ($\|w\|_2 \geq 0.83$, say) the trend is reversed, and the smaller adaptation gain has superior worst-case disturbance rejection performance. This trend reversal illustrates the value of nonlinear induced norm analysis.

Table 4.1 shows the $\mathcal{L}_2$ to $\mathcal{L}_2$ gain with respect to various $\|w\|_2 \leq R$ values for a single quartic V using optimization problem 4.5 and pointwise maximum of 2 quartic V's. From the table, we can see that there is some noticable tightening of the upper bound when the pointwise maximum of 2 quartic V's are used, especially for $\Gamma = 1$. This tightening is at the expense of using more decision variables: optimization using a single quartic V has 162 decision variables, while the pointwise maximum of two quartic V's have 343 decision variables. If a single 6th degree V is used, the upper bound is not as tight as the pointwise maximum of two quartic V's, but uses almost twice as many decision variables (631 vs 343). For example, when $\Gamma = 1$, $R = 1.5$, $\gamma = 0.416$ for the 6th degree V, while $\gamma = 0.412$ for the pointwise maximum of two quartic V's.

Another question one might ask is: starting from the equilibrium point $[z_1, z_2, \theta_e]^T = [0, 0, 0]^T$, what is the $\mathcal{L}_2 \to \mathcal{L}_2$ gain from w to θ_e, and how is this affected by the adaptation gain Γ? Figure 4.6 shows upper and lower on the induced $\mathcal{L}_2 \to \mathcal{L}_2$ gain for $\Gamma = 1, 2$ and 4 at many different values of $\|w\|_2$. The solid lines are the upper bounds obtained by a single quartic V using optimization problem 4.5, while the dashed lines are the upper bounds

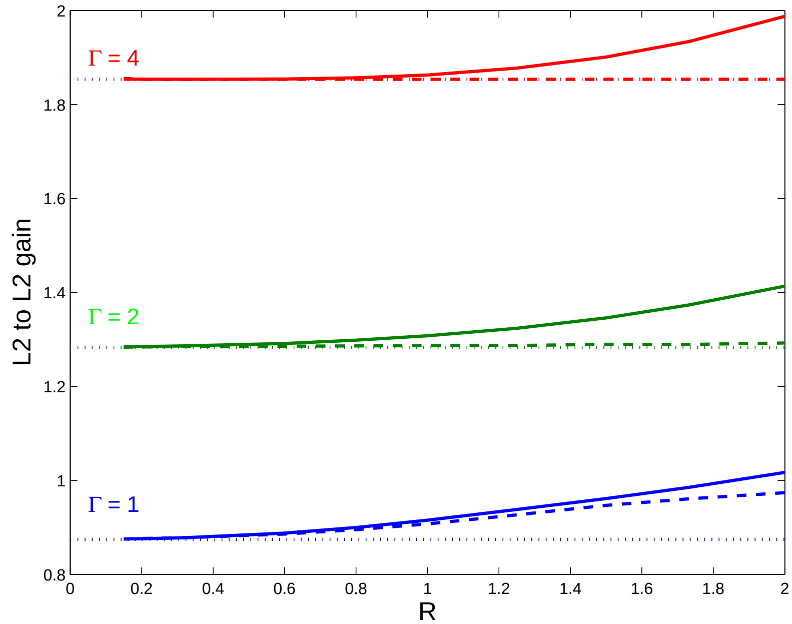

Figure 4.6. Krstić example: $\mathcal{L}_2$ to $\mathcal{L}_2$ gain $(w \to \theta_e)$ for $\Gamma = 1, 2$ and 4

obtained by the pointwise maximum of two quartic V's using optimization problem 4.6 and the dotted lines are the lower bounds. While the effect of adaptation gain on this induced norm is uninteresting, we can see the dramatic improvement in the upper bounds (for $\Gamma = 2$ and 4) when the pointwise maximum of two quartic V's are used.

4.3 Chapter Summary

We have presented the upper bound refinement of the reachable set of a nonlinear system due to a bounded $\mathcal{L}_2$ disturbance. Results from an example showed visible improvement of this refinement and the tightness of the refined upper bound when compared to the lower bound. We also presented a related problem of finding an upper bound of the reachable set due to a disturbance that has both $\mathcal{L}_\infty$ and $\mathcal{L}_2$ bounds. Finally, we presented the technique of analyzing the induced $\mathcal{L}_2 \to \mathcal{L}_2$ gain of a nonlinear system and its refinement through the use of pointwise maximum of polynomials. We illustrate these techniques with an adaptive control system example.

Chapter 5

Synthesis

In this chapter, we present two main ideas on synthesis that utilize sum-of-squares (SOS) programming. The first is on controller synthesis using a Control Lyapunov Function (CLF). Construction of a CLF for a nonlinear system is generally a difficult problem, but once a CLF is found, stabilization of the system is straight-forward. We shall present an optimization problem that searches for CLFs for polynomial systems that are affine in control using SOS programming. We shall also present an optimization problem for searching local CLFs for the same class of nonlinear system when global asymptotic stabilization is not possible. Such a local CLF is also optimized to enlarge a subset of the system's region of attraction using the feedback law derived from the local CLF.

The second idea is on nonlinear observer synthesis using SOS programming. Many control design methods utilize state feedback, but the states are not always known. As such, an observer is needed to estimate the states of the system. For linear systems, Luenberger observers and Kalman filters are widely used. For nonlinear systems, extended Kalman filters, unscented Kalman filters [17], sliding mode observers and geometric methods have been developed. We take the approach of Lyapunov based methods, along the lines proposed by Vidyasagar [39].

5.1 Control Lyapunov Function

Artstein [1] and Sontag [31] showed that for a nonlinear system that is affine in control, the existence of a smooth Control Lyapunov Function (CLF) for the system implies smooth stabilizability for the system. Given a CLF for a nonlinear system, there are several feedback laws that can stabilize the nonlinear system, one of which is given by [31]. Hence, once we have a CLF for the system, stabilization is straight-forward.

On the other hand, construction of the CLF is difficult in general, with the exception of special classes of systems. For example, [10] has shown that for a system that is feedback linearizable, a quadratic CLF can be constructed in the feedback linearized coordinates. In this section, we take the direct approach of searching for CLF through the use of the Positivstellensatz (P-satz) theorem and SOS programming.

Earlier work [14], [15], [16] on controller synthesis using SOS programming involves explicitly searching for a polynomial control law and hence the control law is smooth at the origin. In contrast, the CLF method does not explicitly search for a control law. Moreover, the control law constructed from the CLF might not be a polynomial and is allowed to be non-smooth at the origin, which can be a desirable characteristic [32] because there are examples of systems with $\mathcal{C}^1$ vector fields that cannot be stabilized by a $\mathcal{C}^1$ state feedback controller, but can be stabilized by a controller that is non-smooth at the origin [2].

As with the formulation of stability and performance analysis in the previous two chapters, in the formulation for CLF search, the decision polynomials enter bilinearly, so an algorithm was proposed in [34] that involved a two-way iterative search between the Lyapunov function and the SOS multipliers. With the recent introduction of YALMIP and PENBMI, which allow for bilinear polynomial optimization, we can do away with the two-way iterative search, but as PENBMI is a local bilinear matrix inequality solver, convergence to the global optimum is not guaranteed.

5.1.1 Background

We will first present the single input case and leave the multi-input case till Section 5.1.3. Suppose we are given a system of the form

$$\dot{x} = f(x) + g(x)u \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, f and g are smooth vector fields and $f(0) = 0$.

Definition 5.1. *A function V is a Control Lyapunov Function (CLF) for system (5.1) if $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth, radially unbounded, and positive definite function such that*

$$\inf_{u\in\mathbb{R}} \left\{ \tfrac{\partial V}{\partial x} f(x) + \tfrac{\partial V}{\partial x} g(x)u \right\} < 0 \qquad \forall\, x \neq 0 \,. \tag{5.2}$$

Existence of such a V implies that (5.1) is globally asymptotically stabilizable at the origin.

Further analysis of the LHS of inequality (5.2) reveal that

$$\inf_{u\in\mathbb{R}} \left\{ \tfrac{\partial V}{\partial x} f(x) + \tfrac{\partial V}{\partial x} g(x)u \right\} = \begin{cases} -\infty & \text{when } \tfrac{\partial V}{\partial x} g(x) \neq 0 \\ \tfrac{\partial V}{\partial x} f(x) & \text{when } \tfrac{\partial V}{\partial x} g(x) = 0 \,. \end{cases} \tag{5.3}$$

As a result, V is a CLF if $\frac{\partial V}{\partial x} f(x) < 0$ for all non-zero x such that $\frac{\partial V}{\partial x} g(x) = 0$. With such a CLF, Sontag [31] proposed a feedback law $u = k(x)$, with $k(0) = 0$ that is constructed from the CLF such that the closed loop system is globally asymptotically stable:

$$a(x) := \tfrac{\partial V}{\partial x} f(x) \,, \qquad b(x) := \tfrac{\partial V}{\partial x} g(x) \,, \tag{5.4}$$

$$k(x) := \begin{cases} -\dfrac{a+\sqrt{a^2+b^4}}{b} & \text{when} \quad b \neq 0 \\ 0 & \text{when} \quad b = 0\,, \end{cases} \tag{5.5}$$

$$\frac{dV}{dt} = a(x) + b(x)k(x) = \begin{cases} -\sqrt{a^2+b^4} < 0 & \text{when} \quad b \neq 0 \\ a < 0 & \text{when} \quad b = 0\,. \end{cases} \tag{5.6}$$

Such a $k(x)$ is at least continuous at the origin and smooth everywhere else. Hence, the problem of globally asymptotically stabilizing (5.1) is reduced to finding a CLF for the system, which is a non-trivial problem.

Given a CLF V satisfying (5.2), if we replace V by αV, where $\alpha > 0$, condition (5.2) is still satisfied, but the controller is now α dependent. This α dependence enables us to adjust the magnitude of the control action and the response time taken to converge to the origin, after a CLF is found. The modified controller is:

$$k_\alpha(x) := \begin{cases} -\dfrac{a+\sqrt{a^2+\alpha^2 b^4}}{b} & \text{when} \quad b \neq 0 \\ 0 & \text{when} \quad b = 0\,, \end{cases} \tag{5.7}$$

5.1.2 SOS formulation

In this subsection, we shall show how the definition of a CLF can be formulated as empty set questions so that the P-satz can be applied. This is followed by simplifications to the equations so that SOS programming can be used.

From (5.3), we can see that for a fixed x such that $\frac{\partial V}{\partial x} g(x) \neq 0$, we can make the inequality (5.2) hold by choosing a large value of u of the correct sign. As a result, the crucial place to check is the set of x such that $\frac{\partial V}{\partial x} g(x) = 0$. There, the inequality $\frac{\partial V}{\partial x} f(x) < 0$ must be satisfied, i.e. we want

$$\tfrac{\partial V}{\partial x} f(x) < 0 \;\; \forall x \in \mathbb{R}^n \;\text{ such that }\; \tfrac{\partial V}{\partial x} g(x) = 0,\, x \neq 0\,. \tag{5.8}$$

If we restrict (5.1) to f and g being polynomial vector fields, we can use SOS programming to search for a polynomial CLF V. The condition that V is positive definite and radially unbounded is rewritten as (5.9). Condition (5.8) is rewritten as an empty set condition (5.10) and the search for V is posed as the following feasibility problem:

$$\text{find } V \in \mathcal{R}_n \qquad \text{such that}$$

$$V(x) > 0 \;\; \forall x \in \mathbb{R}^n \backslash \{0\}, \quad V(0) = 0, \quad \text{and} \quad \|V(x)\| \to \infty \text{ as } \|x\| \to \infty, \tag{5.9}$$

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \tfrac{\partial V}{\partial x} g(x) = 0,\, \tfrac{\partial V}{\partial x} f(x) \geq 0,\, x \neq 0 \right\} \text{ is empty}. \tag{5.10}$$

The constraints $x \neq 0$ in (5.9) and (5.10) are equivalent to positive definite polynomials $l_i(x) \neq 0$. Usually $l_i(x)$ is of the form $l_i(x) = \sum_{j=1}^{n} \epsilon_{ij} x_j^2$, where ϵ_{ij} are positive numbers. Condition (5.9) can be reformulated by underbounding V by l_1 (which itself is radially unbounded), and restricting V to be a polynomial with no constant term, i.e. $V(0) = 0$. Our problem is now

$$\text{find} \qquad V \in \mathcal{R}_n \qquad \text{such that } V(0) = 0$$

$$\{x \in \mathbb{R}^n \,|\, V(x) \leq 0,\, l_1(x) \neq 0\} \text{ is empty}, \tag{5.11}$$

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \tfrac{\partial V}{\partial x} g(x) = 0,\, \tfrac{\partial V}{\partial x} f(x) \geq 0,\, l_2(x) \neq 0 \right\} \text{ is empty}. \tag{5.12}$$

Using the P-satz, the above feasibility problem is rewritten as follows:

$$\text{find} \qquad s_0, s_1, s_3, s_4 \in \Sigma_n, \quad V, p_2 \in \mathcal{R}_n, \quad V(0) = 0, \quad k_1, k_2 \in \mathbb{Z}_+$$

such that

$$s_3 - V s_4 + l_1^{2k_1} = 0, \tag{5.13}$$

$$s_0 + s_1 \left[\tfrac{\partial V}{\partial x} f(x) \right] + p_2 \left[\tfrac{\partial V}{\partial x} g(x) \right] + l_2^{2k_2} = 0. \tag{5.14}$$

In order to use SOS programming tools, some simplifications are needed. By choosing

$k_1 = k_2 = 1$, and factoring out a l_1 term in (5.13), and a l_2 term in (5.14), we get the following sufficient conditions:

Optimization Problem 5.1 (Global CLF search):

$$\text{find} \quad s_1 \in \Sigma_n,\ V, p_2 \in \mathcal{R}_n \text{ such that}$$

$$V - l_1 \in \Sigma_n, \tag{5.15}$$

$$-\left\{s_1\left[\tfrac{\partial V}{\partial x} f(x)\right] + p_2\left[\tfrac{\partial V}{\partial x} g(x)\right] + l_2\right\} \in \Sigma_n\,. \tag{5.16}$$

If the above optimization problem is feasible, then V is a CLF for system (5.1).

5.1.3 Multi-input Case

The formulation and feasibility problem presented above can be easily extended to systems with multiple inputs. Consider the following multi-input nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \tag{5.17}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, f and $g_1, \ldots, g_m$ are smooth vector fields and $f(0) = 0$.

For a smooth, radially unbounded, positive definite function V to be a CLF, it must satisfy

$$\tfrac{\partial V}{\partial x} f(x) < 0 \ \ \forall x \in \mathbb{R}^n \setminus \{0\} \text{ such that } \tfrac{\partial V}{\partial x} g_i(x) = 0,\ i = 1, \ldots, m\,. \tag{5.18}$$

Equivalently, the empty set question becomes:

$$\text{Is } \left\{x \in \mathbb{R}^n \;\middle|\; \tfrac{\partial V}{\partial x} g_1(x) = 0, \ldots, \tfrac{\partial V}{\partial x} g_m(x) = 0, \tfrac{\partial V}{\partial x} f(x) \geq 0,\ x \neq 0\right\} \text{ empty?} \tag{5.19}$$

Applying P-satz to (5.19) and using the same simplifications as in (5.14), we get

$$-\left\{s_1\left[\tfrac{\partial V}{\partial x} f(x)\right] + \sum_{i=1}^{m} p_{2i}\left[\tfrac{\partial V}{\partial x} g_i(x)\right] + l_2\right\} \in \Sigma_n\,. \tag{5.20}$$

For multi-input systems, we just need to replace (5.16) with (5.20) in the optimization. The number of SOS constraints still remains the same, but we need to search over polynomials $p_{21}, \ldots, p_{2m}$, instead of just p_2.

The multi-input state feedback controller is constructed as follows [31]:

$$a(x) := \tfrac{\partial V}{\partial x} f(x) \ , \qquad b_i(x) := \tfrac{\partial V}{\partial x} g_i(x) \ , \qquad \beta(x) := \sum_{i=1}^{m} b_i^2(x) \ , \tag{5.21}$$

$$k_i(x) := \begin{cases} -b_i \dfrac{a + \sqrt{a^2 + \beta^2}}{\beta} & \text{when} \quad \beta \neq 0 \\ 0 & \text{when} \quad \beta = 0 \ . \end{cases} \tag{5.22}$$

Again, we can use a parameter, $\alpha > 0$, to adjust the magnitude of the control action:

$$k_{i\alpha}(x) := \begin{cases} -b_i \dfrac{a + \sqrt{a^2 + \alpha^2 \beta^2}}{\beta} & \text{when} \quad \beta \neq 0 \\ 0 & \text{when} \quad \beta = 0 \ . \end{cases} \tag{5.23}$$

5.1.4 Examples

5.1.4.1 Example 1 - A bilinear system

The following 2nd order bilinear system is taken from [37], which has shown that this system can be globally asymptotically stabilized by an appropriate mixing of the stabilizing controllers for the slow and fast subsystems. We shall use feasibility problem 5.1 to find a stabilizing controller for this system without exploiting such knowledge.

$$\begin{aligned} \dot{x}_1 &= (3x_1 + 4x_2)u \\ \dot{x}_2 &= (-20x_1 + 10x_2)u \end{aligned} \tag{5.24}$$

Also, for this example, the veracity of the solution from our optimization can be easily checked with some simple analysis using Linear Matrix Inequalities (LMIs), which we shall present below.

Given a bilinear system of the form

$$\dot{x} = (Lx)u \tag{5.25}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $L \in \mathbb{R}^{n\times n}$, the only way that (5.2) can be satisfied is by finding a V such that the set $\left\{x \in \mathbb{R}^n \mid \frac{\partial V}{\partial x} Lx = 0\right\}$ is empty, so either $\frac{\partial V}{\partial x} Lx > 0$ or $\frac{\partial V}{\partial x} Lx < 0$ for all $x \neq 0$. We can take this analysis further by considering quadratic $V = \frac{1}{2}x^T Px$, with $P \succ 0$.

Proposition 5.1. *A quadratic V is a CLF for system* (5.25) *if either one of the following SDP problem is feasible:*

Find $P \in \mathbb{R}^{n\times n}$ such that		*Find $P \in \mathbb{R}^{n\times n}$ such that*
$P \succ 0$ *and* $PL \succ 0$	*or*	$P \succ 0$ *and* $PL \prec 0$

Applying Proposition 5.1 to our example, let $V = \frac{1}{2}x^T Px$, where

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \succ 0\,. \tag{5.26}$$

Suppose we choose $\frac{\partial V}{\partial x} g(x) > 0$ for all $x \neq 0$. This condition can be reduced to an LMI:

$$\frac{\partial V}{\partial x} g(x) = (3P_{11} - 20P_{12})x_1^2 + (4P_{11} - 20P_{22} + 13P_{12})x_1x_2 + (10P_{22} + 4P_{12})x_2^2 =: x^T Mx > 0$$

$$\Leftrightarrow \qquad M := \begin{bmatrix} 3P_{11} - 20P_{12} & 2P_{11} - 10P_{22} + 6.5P_{12} \\ 2P_{11} - 10P_{22} + 6.5P_{12} & 10P_{22} + 4P_{12} \end{bmatrix} \succ 0\,. \tag{5.27}$$

If we can find a V such that (5.26) and (5.27) are satisfied, then V is a CLF. We will verify that the CLFs for this system obtained from our optimization satisfy LMIs (5.26) and (5.27). As a side note, $V = \frac{1}{2}(x_1^2 + x_2^2)$ is not a CLF because it does not satisfy (5.27).

We set the degrees of V, s_1 and p_2 to search over to be 2, 4 and 2 respectively. The resulting CLF from the feasibility problem 5.1 is $V = 3.01x_1^2 - 0.143x_1x_2 + 1.00x_2^2$, which can be easily verified that it satisfies (5.26) and (5.27).

5.1.4.2 Example 2 - A multi-input example

This 3-state system, which has a 2-dimensional locally stable manifold, is from example 3 of [3]. They showed that the system can be globally stabilized by forcing the dynamics onto that stable switching manifold and hence the control action exhibits chattering about the switching surface. We shall use SOS programming to search for a CLF for this system.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} -x_1 + x_2^2 \\ -2x_2 \\ 3x_3 + x_2^3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 + x_3^2 \\ 1 + 4x_1^2 \end{bmatrix} u_1 + \begin{bmatrix} 5x_1 \\ 1 - x_2^2 \\ 3 \end{bmatrix} u_2 \tag{5.28}$$

The degrees of V, s_1 and p_{2i} are chosen to be 2, 0 and [1, 1] respectively. We performed the optimization with constraints (5.15) and (5.20) and obtained a feasible result, i.e. we found a CLF for this system. Using the feedback formula (5.22), a simulation with initial conditions $x_0 = [-1, 0.8, 1]^T$ was performed and the results are shown in Figure 5.1. Compared to [3], our response times are faster, but use more control action. We could have traded off response time vs control action using the α dependent controller (5.23).

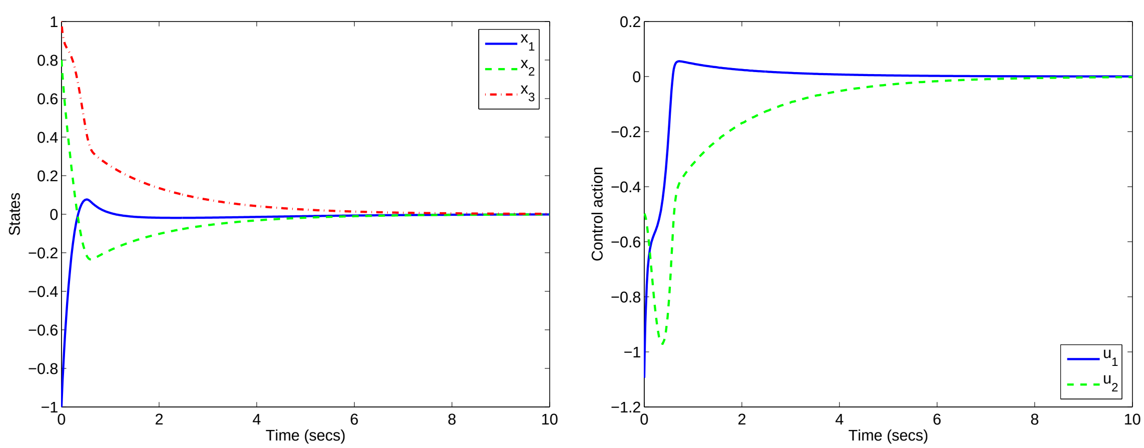

Figure 5.1. Multi-input CLF example. Left: States, Right: Control action

5.2 Local Control Lyapunov Function

When system (5.1) cannot be globally asymptotically stabilized, we ask whether we can locally stabilize the system and how large can we make its region of attraction. In this section, we shall present an optimization problem for searching a local CLF.

5.2.1 SOS Formulation

When V, a candidate CLF, fails to satisfy (5.2), it is because there are points x such that $\frac{\partial V}{\partial x}g(x) = 0$ and $\frac{\partial V}{\partial x}f(x) \geq 0$. If a system cannot be globally stabilized, another CLF-based approach is to find a compact set that excludes such points.

We want to find a level set $\Omega = \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\}$ such that $\forall x \in \Omega \setminus \{0\}$ and $\frac{\partial V}{\partial x}g(x) = 0$, we have $\frac{\partial V}{\partial x}f(x) < 0$. The level set Ω is a region of attraction for the closed loop system when we use Sontag's feedback law (5.5). This is because for all $x \in \Omega \setminus \{0\}$ such that $b = \frac{\partial V}{\partial x}g(x) \neq 0$, we have $\frac{dV}{dt} = -\sqrt{a^2 + b^4} < 0$ and when $b = 0$, $\frac{dV}{dt} = a < 0$.

As in previous chapters, to enlarge Ω, pick a positive definite p. Define a variable sized region $P_\beta := \{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\}$ such that $P_\beta \subseteq \Omega$. By maximizing β, we are enlarging P_β and Ω. These two conditions result in the following set containment constraints:

$$\left\{x \in \mathbb{R}^n \,\middle|\, V(x) \leq 1,\, \frac{\partial V}{\partial x}g(x) = 0\right\} \setminus \{0\} \subseteq \left\{x \in \mathbb{R}^n \,\middle|\, \frac{\partial V}{\partial x}g(x) = 0,\, \frac{\partial V}{\partial x}f(x) < 0\right\}, \quad (5.29)$$

$$\{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\}. \quad (5.30)$$

The equivalent empty set questions of (5.29) and (5.30) become:

$$\text{Are } \left\{x \in \mathbb{R}^n \,\middle|\, V(x) \leq 1,\, \frac{\partial V}{\partial x}g(x) = 0,\, \frac{\partial V}{\partial x}f(x) \geq 0,\, x \neq 0\right\} \text{ and} \quad (5.31)$$

$$\{x \in \mathbb{R}^n \,|\, p(x) \leq \beta, V(x) \geq 1, V(x) \neq 1\} \text{ empty?} \quad (5.32)$$

Again, using a positive definite, SOS polynomial $l_2(x)$ to replace the non-polynomial constraint $x \neq 0$ in (5.31) and applying the P-satz to (5.31) and (5.32), we have

$$s_0 + s_1(1-V) + s_2\left[\tfrac{\partial V}{\partial x} f(x)\right] + s_3(1-V)\left[\tfrac{\partial V}{\partial x} f(x)\right] + p_4\left[\tfrac{\partial V}{\partial x} g(x)\right] + l_2^{2k_2} = 0 \tag{5.33}$$

$$s_5 + s_6(\beta - p) + s_7(V-1) + s_8(\beta - p)(V-1) + (V-1)^{2k_3} = 0 \tag{5.34}$$

By choosing $s_3 = 0$ and $k_2 = 1$, and factoring out a l_2 term, we simplify the constraint (5.33) into sufficient condition

$$-\left\{s_1(1-V) + s_2\tfrac{\partial V}{\partial x} f(x) + p_4\tfrac{\partial V}{\partial x} g(x) + l_2\right\} \in \Sigma_n\,. \tag{5.35}$$

Equation (5.34) has a $(V-1)^{2k_3}$ term which cannot be optimized using SOS programming, so we cast this constraint as an $\mathcal{S}$-procedure by setting $s_5 = s_6 = 0$, $k_3 = 1$, and factoring out a $(V-1)$ term.

With these simplifications to (5.33) and (5.34), a sufficient condition for a local CLF is formulated as the following optimization problem:

Optimization Problem 5.2 (Local CLF optimization):

$$\max\ \beta \qquad \text{over}\ s_1, s_2, s_8 \in \Sigma_n,\quad V, p_4 \in \mathcal{R}_n,\ V(0) = 0$$

such that

$$V - l_1 \in \Sigma_n \tag{5.36}$$

$$-\Big((\beta - p)s_8 + (V-1)\Big) \in \Sigma_n \tag{5.37}$$

$$-\left\{s_1(1-V) + s_2\tfrac{\partial V}{\partial x} f(x) + p_4\tfrac{\partial V}{\partial x} g(x) + l_2\right\} \in \Sigma_n \tag{5.38}$$

Again, the constraints (5.37) and (5.38) are bilinear in the decision polynomials. We can easily extend this formulation for local CLFs to the multi-input case, using the ideas in Section 5.1.3.

5.2.2 Example

This example is the taken from [15]. Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_1 + \tfrac{1}{6}x_1^3 - u \, . \end{aligned} \tag{5.39}$$

We shall analytically show that quadratic V's do not meet the (global) CLF condition (5.2) for this system. Define $V := \frac{1}{2}x^T P x$, where P is a positive definite symmetric matrix:

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

We look at the sign of $\frac{\partial V}{\partial x} f(x)$ when $\frac{\partial V}{\partial x} g(x) = 0$:

$$\tfrac{\partial V}{\partial x} f(x) = x^T P f = (P_{12}x_1 + P_{22}x_2)(-x_1 + \tfrac{1}{6}x_1^3) \tag{5.40}$$

$$\tfrac{\partial V}{\partial x} g(x) = x^T P g = (P_{11} - P_{12})x_1 + (P_{12} - P_{22})x_2 = 0 \tag{5.41}$$

Since (5.41) is a linear equation, we can solve for x_2 and substitute it into (5.40):

$$x_2 = m x_1 \qquad \text{where} \qquad m := \frac{P_{11} - P_{12}}{P_{22} - P_{12}}$$

$$\begin{aligned} \tfrac{\partial V}{\partial x} f(x) &= (P_{12}x_1 + P_{22}x_2)(-x_1 + \tfrac{1}{6}x_1^3) \\ &= (P_{12} + P_{22}m)(-x_1^2 + \tfrac{1}{6}x_1^4) \end{aligned} \tag{5.42}$$

The first term in (5.42) is a constant, which could be positive or negative, depending on the choice of P. The second term in (5.42) is a quartic function in x_1 and the roots of this function are $0, 0, -\sqrt{6}, \sqrt{6}$. The interval $(-\sqrt{6}, \sqrt{6})$ is of opposite sign to the set $(-\infty, -\sqrt{6}) \bigcup (\sqrt{6}, \infty)$, regardless of how we chose the entries of P. As such, it is not possible for (5.42) to be negative definite, and hence this system is not globally stabilizable for a quadratic V. When the global CLF optimization problem 5.1 is used to search for a global quadratic CLF V, we get infeasible result, which is expected. We can also try using higher degree V's, but instead, we will take the approach of finding a local CLF.

We use optimization problem 5.2 to find a local CLF for this system by setting the degrees of V, s_1, s_2, s_8 and p_4 to be 2, 2, 0, 0 and 3 respectively. The level set that we are interested in enlarging is $P_\beta := \{x \in \mathbb{R}^2 \,|\, p(x) \leq \beta\}$, where $p(x) = \frac{1}{6}x_1^2 + \frac{1}{6}x_1x_2 + \frac{1}{12}x_2^2$. For a quadratic V, the region where (5.42) is negative is on the line segment $x_2 = mx_1$, for $x_1 \in (-\sqrt{6}, \sqrt{6})$ and $x_2 \in (-m\sqrt{6}, m\sqrt{6})$. Hence, this system has a semi-global stabilization property, where β can be made arbitrarily large by optimizing the coefficients of V to obtain large m values, and so we set an upper bound of $\beta = 100$ for this example. After optimization, $\beta = 100$ is indeed obtained and the CLF is $V = 0.912x_1^2+1.015x_1x_2+0.538x_2^2$. In comparison with our previously published result [34], where $\beta = 38.37$ after 50 iterations, our new optimization utilizing PENBMI gives a much larger β value without resorting to "V-S" iterations.

Figure 5.2 shows the level set $\{x \in \mathbb{R}^2 \,|\, V(x) \leq 1\}$ which is a region of attraction for this system when we use the resulting local CLF and the corresponding feedback law. The line segment shows both the set $\left\{x \in \mathbb{R}^n \,|\, \frac{\partial V}{\partial x}g(x) = 0,\ \frac{\partial V}{\partial x}f(x) < 0\right\}$ and our region of attraction stay within this line segment in this direction.

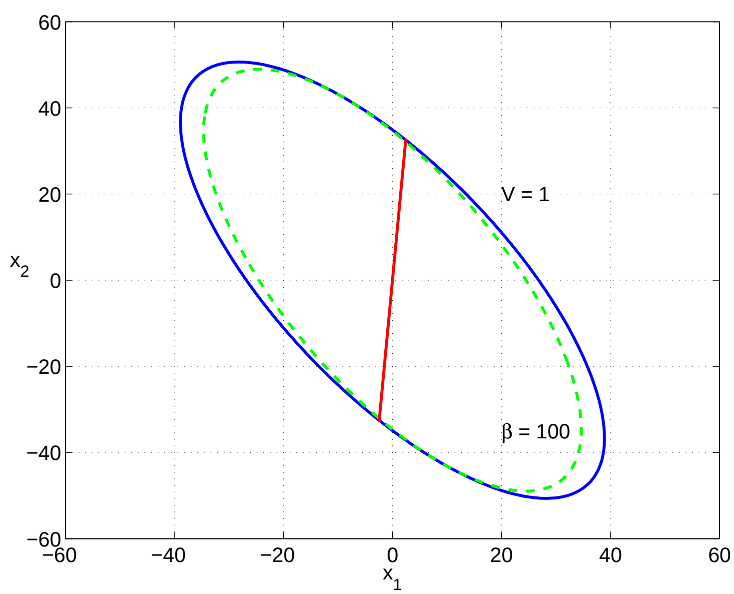

Figure 5.2. CDC'03 example: Closed loop system's region of attraction

5.3 Nonlinear Observers

In this section, we will derive sufficient conditions for finding polynomial observers for polynomial systems using SOS programming. We develop our optimization problem of searching for the observer using the concept of weak detectability, as proposed by Vidyasagar [39]. Background material from [39] will be presented in the next subsection, 5.3.1.

5.3.1 Background

Consider a nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{5.43}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^{n_y}$ and $u(t) \in \mathbb{R}^{n_u}$. In addition, this system is assumed to satisfy the following conditions:

1. f is continuously differentiable and $f(0,0) = 0$,

2. there are constants α and c such that

$$\left|\left|\frac{\partial f}{\partial x}\right|\right| \leq \alpha \text{ and } \left|\left|\frac{\partial f}{\partial u}\right|\right| \leq \alpha \quad \forall x \in P_{\mathrm{x}}, \forall u \in P_{\mathrm{u}},$$

3. h is continuous and $h(0) = 0$.

Here, P_{x} and P_{u} denote some closed regions centered at the origin.

Definition 5.2. *The system* (5.43) *is said to be weakly detectable if one can find a function* $L : \mathbb{R}^n \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}^n$ *and a function* $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ *such that*

1. *L is continuous and $L(0,0,0) = 0$,*

2. there exists class $\mathcal{K}$ functions ψ_1, ψ_2, ψ_3 such that

$$\psi_1(\|x-z\|) \le V(x,z) \le \psi_2(\|x-z\|) \qquad \forall x \in P_{\mathrm{x}},\ \forall z \in P_{\mathrm{z}} \tag{5.44}$$

$$\tfrac{\partial V}{\partial x} f(x,u) + \tfrac{\partial V}{\partial z} L(z,h(x),u) \le -\psi_3(\|x-z\|) \qquad \forall x \in P_{\mathrm{x}},\ \forall z \in P_{\mathrm{z}},\ \forall u \in P_{\mathrm{u}} \tag{5.45}$$

According to Definition 5.2, if the system (5.43) is weakly detectable, then we can set up a weak detector

$$\dot{z} = L(z,y,u)\ \ (= L(z,h(x),u)) \tag{5.46}$$

for system (5.43). If $u(t)$ always stays in P_{u} and if the solution trajectories $x(t)$ and $z(t)$ do not leave P_{x} and P_{z} respectively, then $x(t) - z(t) \to 0$ as $t \to \infty$. However, if these conditions on $x(t)$, $z(t)$ and $u(t)$ are not satisfied, there is no guarantee that $x(t) - z(t)$ will tend towards 0. If $P_{\mathrm{x}} = \mathbb{R}^n$, $P_{\mathrm{z}} = \mathbb{R}^n$ and $P_{\mathrm{u}} = \mathbb{R}^{n_u}$, then L is said to be a *global detector* for system (5.43).

Consider the special case of systems of the form

$$\begin{aligned} \dot{x} &= f_1(x) + f_2(h(x),u)\ , \qquad && f_1(0)=0, \quad f_2(0,0)=0 \\ y &= h(x)\ , && h(0) = 0\ . \end{aligned} \tag{5.47}$$

Lemma 5.1. *Given system* (5.47)*, if we can find a function $\tilde{V} : \mathbb{R}^n \to \mathbb{R}_+$, a continuous function $\tilde{L} : \mathbb{R}^n \times \mathbb{R}^{n_y} \to \mathbb{R}^n$ with $\tilde{L}(0,0) = 0$, and class $\mathcal{K}$ functions ψ_4, ψ_5, ψ_6 such that*

$$\psi_4(\|x-z\|) \le \tilde{V}(x-z) \le \psi_5(\|x-z\|) \qquad \forall x \in P_{\mathrm{x}},\ \forall z \in P_{\mathrm{z}} \tag{5.48}$$

$$\tfrac{\partial \tilde{V}}{\partial (x-z)} \left[f_1(x) - \tilde{L}(z,h(x)) \right] \le -\psi_6(\|x-z\|) \qquad \forall x \in P_{\mathrm{x}},\ \forall z \in P_{\mathrm{z}}, \tag{5.49}$$

then the system is weakly detectable.

Proof. Define $V(x,z) := \tilde{V}(x-z)$, then (5.48) is equivalent to (5.44). Also define $L(z,y,u) := \tilde{L}(z,y) + f_2(y,u)$, then

$$\frac{\partial \tilde{V}}{\partial(x-z)}\left[f_1(x)-\tilde{L}(z,h(x))\right] = \frac{\partial \tilde{V}}{\partial(x-z)}\left[(f_1(x)+f_2(h(x),u)) - (\tilde{L}(z,h(x))+f_2(h(x),u))\right]$$

$$= \frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial z} L(z,h(x),u)\,,$$

so (5.49) is a special case of (5.45), where it holds for all $x \in P_{\mathrm{x}}$, $z \in P_{\mathrm{z}}$, and $u \in \mathbb{R}^{n_u}$. □

Lemma 5.1 states that if we have a system of the form (5.47), then we only need to find an u independent weak detector $\tilde{L}(z,y)$. This is because f_2 is a function of y and u, which are known and can be measured, and not a function of x (unknown), so if the function f_2 is known exactly, it can be duplicated in L, i.e. $L(z,y,u) = \tilde{L}(z,y) + f_2(y,u)$, so that the effects of $f_2(y,u)$ are canceled out in $\dot{x} - \dot{z}$.

Definition 5.3. *The system* (5.43) *is said to be stabilizable if one can find a function* $k : \mathbb{R}^n \to \mathbb{R}^{n_u}$ *with the following properties:*

1. *k is continuously differentiable with $k(0) = 0$, and β is a class $\mathcal{K}$ function such that*

$$\left|\left|\frac{\partial k}{\partial x}\right|\right| \le \beta(\|x\|), \quad \forall x \in P_{\mathrm{x}}\,,$$

2. *$x = 0$ is an asymptotically stable equilibrium point of $\dot{x} = f(x, k(x))$.*

The function k is called a stabilizing control law for (5.43).

Theorem 5.1. *[39][Theorem 3.1]. Consider the system* (5.43). *If it is stabilizable and weakly detectable, then $x = 0$ and $z = 0$ is an asymptotically stable equilibrium point of the system*

$$\begin{aligned} \dot{x} &= f(x,k(z)) \qquad , y = h(x) \\ \dot{z} &= L(z,y,k(z)) \end{aligned} \tag{5.50}$$

Theorem 5.1 states that if the system (5.43) is stabilized by the control law $u(t) = k(x(t))$, then it is also stabilized by the control law $u(t) = k(z(t))$, where $z(t)$ is the output of the weak detector for $x(t)$. However, this theorem only states that the equilibrium point is locally asymptotically stable, but it does not quantify the region of attraction. Of course, we can use our method of enlarging a provable region of attraction as presented in Section 3.1 for the system (5.50).

5.3.2 SOS formulation

If we restrict the system (5.47) to polynomial vector fields and are searching for a polynomial weak detector $\tilde{L}$, we can use SOS programming in the synthesis of $\tilde{L}$.

First, note that in (5.44) – (5.45) and (5.48) – (5.49), ψ_i are class $\mathcal{K}$ functions that are not necessarily polynomials. We will replace these class $\mathcal{K}$ functions with positive definite polynomials with the help of the lemma below [18, Lemma 4.3].

Lemma 5.2. *Let $W : D \to \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that*

$$\alpha_1(\|x\|) \le W(x) \le \alpha_2(\|x\|) \tag{5.51}$$

for all $x \in B_r$. If $D = R^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the foregoing inequality will hold for all $x \in \mathbb{R}^n$. Moreover, if $W(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$.

Lemma 5.3. *Let $l(x) = \sum_{j=1}^{n} \epsilon_j x_j^{2k}$. Then for any $q \in \Sigma_n$ with $q(0) = 0$, $k \in \mathbb{Z}_+$ and $\epsilon_j > 0$, there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\forall x \in \mathbb{R}^n$,*

$$\alpha_1(\|x\|) \le l(x) + q(x) \le \alpha_2(\|x\|) \tag{5.52}$$

Proof. This lemma is a direct consequence of Lemma 5.2. The function $W(x) := l(x)+q(x)$ is continuous and positive definite on the domain $D = \mathbb{R}^n$. Moreover, since l is radially unbounded, W is radially unbounded as well. □

By Lemma 5.3, the class $\mathcal{K}$ functions ψ_i in (5.44) – (5.45) and (5.48) – (5.49) can be replaced by positive definite polynomials W_i of the form $W_i(x-z) = l_i(x-z) + \hat{W}_i(x-z)$, where $l_i(x-z) = \sum_{j=1}^{n} \epsilon_{ij}(x_j - z_j)^2$, and $\hat{W}_i \in \Sigma_n$ with $\hat{W}_i(0) = 0$. For example in (5.48),

$$\psi_4(\|x-z\|) \leq \tilde{V}(x-z) \leq \psi_5(\|x-z\|) \qquad \forall x \in P_{\mathrm{x}},\ \forall z \in P_{\mathrm{z}}$$

can be replaced with

$$W_4(x-z) \leq \tilde{V}(x-z) \leq W_5(x-z) \qquad \forall x \in P_{\mathrm{x}},\ \forall z \in P_{\mathrm{z}} \tag{5.53}$$

because there exists class $\mathcal{K}$ functions $\psi_4(\|x-z\|)$ and $\psi_5(\|x-z\|)$ such that

$$\psi_4(\|x-z\|) \leq W_4(x-z) \leq \tilde{V}(x-z) \leq W_5(x-z) \leq \psi_5(\|x-z\|) \qquad \forall x \in P_{\mathrm{x}}, \forall z \in P_{\mathrm{z}} \tag{5.54}$$

By replacing the class $\mathcal{K}$ functions in (5.48) – (5.49) with positive definite functions, we have the following sufficient conditions for Lemma 5.1:

If there exists a n vector-valued polynomial $\tilde{L} \in \mathcal{R}_{n+n_y}$ with $\tilde{L}(0,0) = 0$, a polynomial $\tilde{V} \in \mathcal{R}_n$, and positive definite polynomials $W_4, W_5, W_6 \in \mathcal{R}_n$ such that

$$W_4(x-z) \leq \tilde{V}(x-z) \leq W_5(x-z) \qquad \forall x \in P_{\mathrm{x}},\ \forall z \in P_{\mathrm{z}} \tag{5.55}$$

$$\tfrac{\partial \tilde{V}}{\partial(x-z)} \left[f_1(x) - \tilde{L}(z, h(x)) \right] \leq -W_6(x-z) \qquad \forall x \in P_{\mathrm{x}},\ \forall z \in P_{\mathrm{z}}, \tag{5.56}$$

then the system $\dot{z} = L(z,y,u) = \tilde{L}(z,y) + f_2(y,u)$ is a weak detector for the system (5.47).

We usually search for $\tilde{L}$ of the form $\tilde{L} := L_M L_V$ where L_M is a matrix of real numbers with n rows, and L_V is a vector of polynomials formed from the multiplicative monoid of

(z, y). For example, if the system (5.47) has $n = 2$, and $n_y = 1$, and we are searching for $\tilde{L}$ with linear and cubic terms only, then

$$L_V = [z_1,\, z_2,\, y,\, z_1^3,\, z_1^2 z_2,\, z_1 z_2^2,\, z_2^3,\, z_1^2 y, z_1 z_2 y,\, z_2^2 y,\, z_1 y^2,\, z_2 y^2,\, y^3]^T. \tag{5.57}$$

5.3.2.1 Global detectors

Now, we shall show the SOS formulation for the simplest case where we are synthesizing a global detector for system (5.47). For global detectors, the quantifiers in constraints (5.55) and (5.56) should hold for all x, z, i.e. $P_{\mathrm{x}} = \mathbb{R}^n$ and $P_{\mathrm{z}} = \mathbb{R}^n$. Constraint (5.55) is automatically satisfied if we restrict the search of $\tilde{V}$ to positive definite polynomials, as W_4 and W_5 can be chosen to be $\tilde{V}$.

The system $\dot{z} = L(z, y, u) = \tilde{L}(z, y) + f_2(y, u)$ is a global detector for system (5.47) if the following optimization is feasible:

Optimization Problem 5.3 (Global detector synthesis):

$$\begin{aligned} \text{Find} \quad & \hat{V} \in \Sigma_n,\ \hat{V}(0) = 0,\ \epsilon_{vj} > 0,\ \hat{W}_6 \in \Sigma_n,\ \hat{W}_6(0) = 0,\ \epsilon_{6j} > 0, \\ & \tilde{L} \in \mathcal{R}_{n+n_y},\ \tilde{L}(0,0) = 0 \end{aligned}$$

such that with

$$\begin{aligned} &\tilde{V}(x - z) := \hat{V}(x - z) + \sum_{j=1}^{n} \epsilon_{vj}(x_j - z_j)^2, \\ &W_6(x - z) := \hat{W}_6(x - z) + \sum_{j=1}^{n} \epsilon_{6j}(x_j - z_j)^2, \\ &-\left\{ \tfrac{\partial \tilde{V}}{\partial (x-z)} \left[f_1(x) - \tilde{L}(z, h(x)) \right] + W_6(x - z) \right\} \in \Sigma_{2n}\,. \end{aligned} \tag{5.58}$$

Note that this optimization problem is bilinear in the decision polynomials $\tilde{V}$ and $\tilde{L}$.

5.3.2.2 Weak detectors

We now focus on the non-global case of finding a weak detector for system (5.47). For simplicity in SOS formulation, at the expense of being more restrictive, we will require that (5.55) hold for all $x, z \in \mathbb{R}^n$, instead of for all $x \in P_{\mathrm{x}}$, and $z \in P_{\mathrm{z}}$. Due to this simplification, constraint (5.55) is automatically satisfied if we restrict the search of $\tilde{V}$ to positive definite polynomials.

We want constraint (5.56) to hold for as large a region as possible, so that the weak detector works in this region. Let the closed sets $P_x := \{x \in \mathbb{R}^n \,|\, p_x(x) \le c\}$ and $P_z := \{z \in \mathbb{R}^n \,|\, p_z(z) \le c\}$, where p_{x} and p_{z} are positive definite polynomials chosen by the user. The resulting set containment condition is

$$\{(x,z) \,|\, p_x(x) \le c, p_z(z) \le c\} \subseteq \left\{(x,z) \,\middle|\, -\tfrac{\partial \tilde{V}}{\partial (x-z)} \left[f_1(x) - \tilde{L}(z, h(x))\right] - W_6(x-z) \ge 0 \right\}. \tag{5.59}$$

Applying generalized $\mathcal{S}$-procedure to (5.59), we have the following maximization problem:

Optimization Problem 5.4 (Weak detector synthesis):

$$\max\ c \qquad \text{over} \quad \hat{V} \in \Sigma_n,\ \hat{V}(0) = 0,\ \epsilon_{vj} > 0,\ \hat{W}_6 \in \Sigma_n,\ \hat{W}_6(0) = 0,\ \epsilon_{6j} > 0$$

$$\tilde{L} \in \mathcal{R}_{n+n_y},\ \tilde{L}(0,0) = 0,\ s_1, s_2 \in \Sigma_{2n},$$

such that with

$$\tilde{V}(x-z) := \hat{V}(x-z) + \sum_{j=1}^{n} \epsilon_{vj}(x_j - z_j)^2,$$

$$W_6(x-z) := \hat{W}_6(x-z) + \sum_{j=1}^{n} \epsilon_{6j}(x_j - z_j)^2,$$

$$-\left\{\tfrac{\partial \tilde{V}}{\partial (x-z)} \left[f_1(x) - \tilde{L}(z, h(x))\right] + W_6(x-z)\right\}$$
$$- s_1(x,z)(c - p_{\mathrm{x}}(x)) - s_2(x,z)(c - p_{\mathrm{z}}(z)) \in \Sigma_{2n}\,. \tag{5.60}$$

5.3.3 Examples

5.3.3.1 Example 1 - Duffing Equations

The Duffing equations are as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 \\ y &= x_1 + \tfrac{1}{2}x_2. \end{aligned} \tag{5.61}$$

This system is autonomous and has 3 equilibrium points: $(0,0)$, a saddle, and $(\pm 1, 0)$, which are centers. It is known to have a conservative energy function $E = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$, so the system's trajectories are periodic. Depending on the initial conditions, the system will exhibit one of the 3 types of orbits as shown in Figure 5.3.

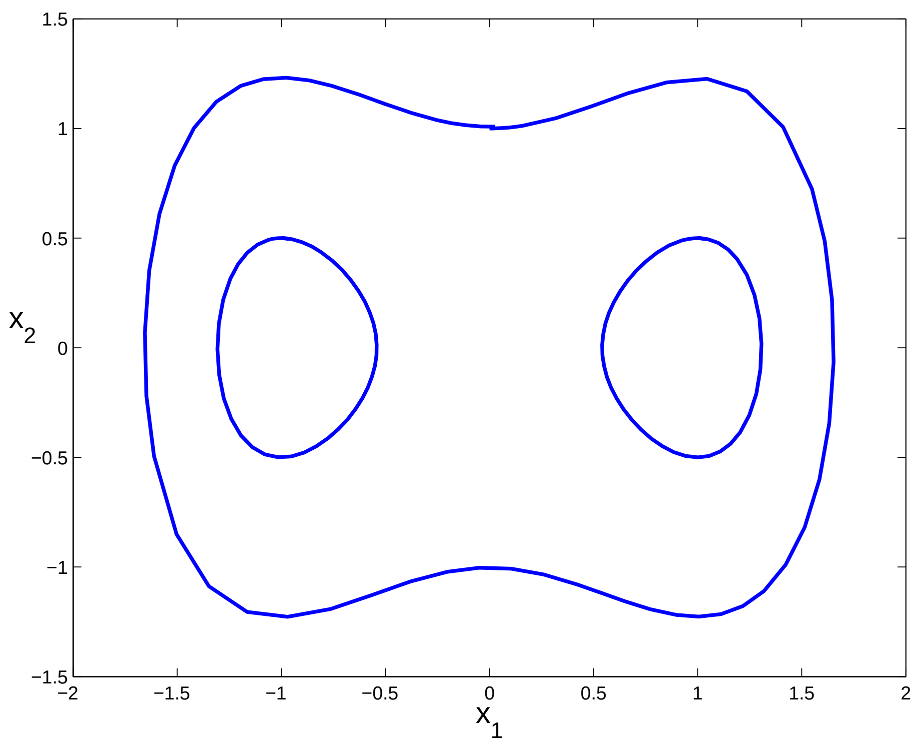

Figure 5.3. Duffing Equations: Periodic orbits

We are interested in synthesizing a global nonlinear detector using SOS programming. Since the system is autonomous, the feasibility problem 5.3 is applicable here. We choose to search over quartic $\tilde{V}(x-z)$, quadratic $W_6(x-z)$ and $\tilde{L}$ with only linear and cubic terms in z and y, so $L = \tilde{L} = L_M L_V$, where L_V is as shown in (5.57) and L_M is a $\mathbb{R}^{2\times 13}$ matrix.

We obtained a feasible result and implemented the global detector in SIMULINK. The simulation results are shown in Figure 5.4. The estimation error is defined as $e := x - z$. The first three rows are results for the system starting from initial conditions representative of the three types of orbits, with the detector's initial states at the origin. The last row is to demonstrate the global nature of the detector by starting the detector states very far away (at $(-5, -5)$) from the system's initial conditions (at $(0, 1)$). In all four cases, the estimation error converges to zero within 2 secs, however, the transient behavior is poor, with large overshoot due to the aggressive nature of the global detector. Unfortunately, as presented, feasibility problem 5.3 does not have a performance measure to penalize the overshoot. In comparison, [20] designed a locally convergent nonlinear observer using backstepping for the same problem, but their estimation error converges to zero in about 5 seconds, with a much smaller overshoot.

5.3.3.2 Example 2 - State Feedback

Once again, we revisit the state feedback example from [15]. Suppose we do not have access to both states of the system and have to estimate them in order to utilize the state feedback controllers already designed in [15], [16], [34] and Section in 5.2.2.

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_1 + \tfrac{1}{6}x_1^3 - u \\ y &= x_2 \,. \end{aligned} \tag{5.62}$$

Since the system is of the form (5.47) and is time-invariant, we shall use the optimization

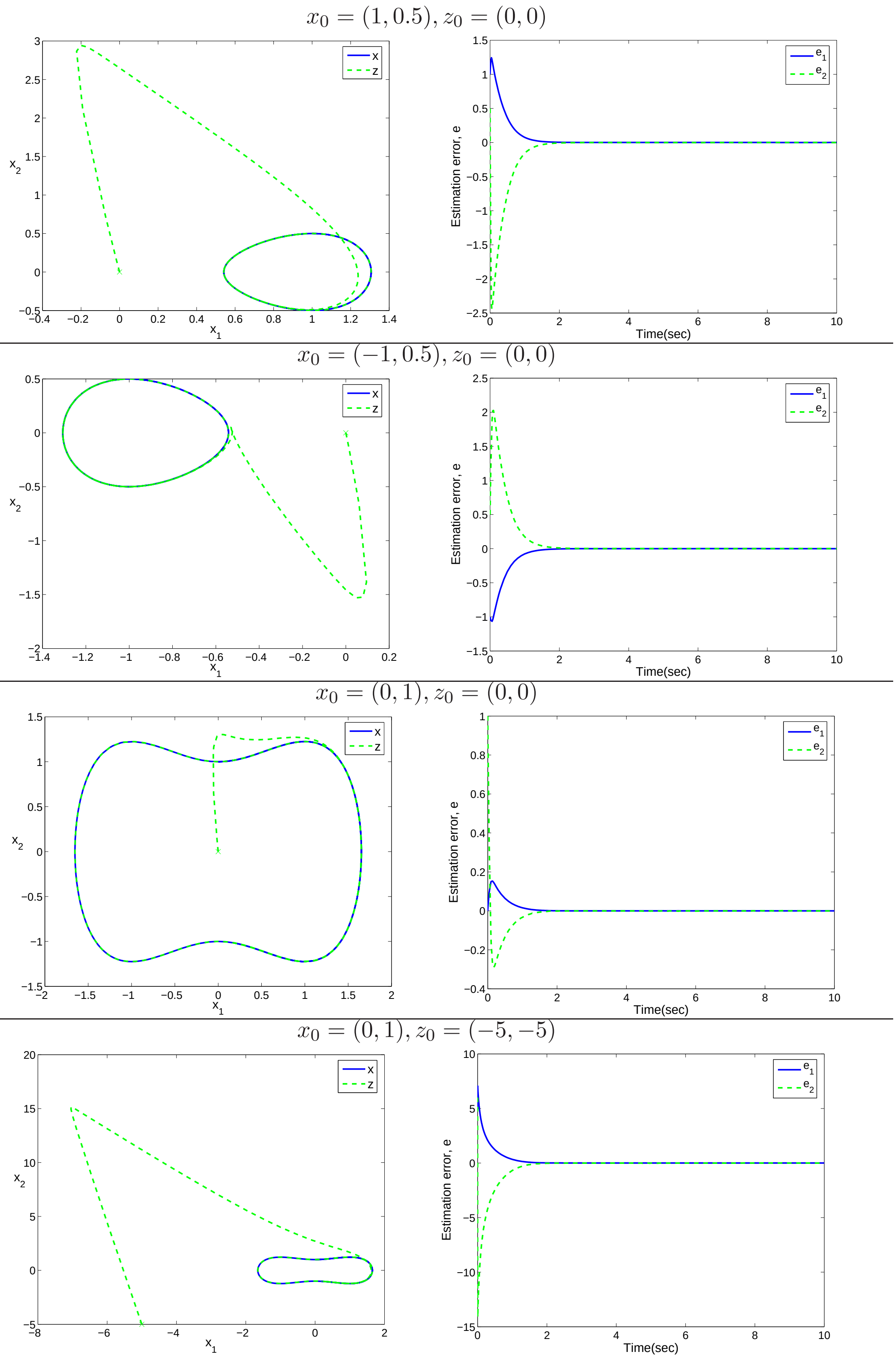

Figure 5.4. Duffing Equations: Left: Estimate vs actual states; Right: Estimation errors

problem 5.4 to search for a weak detector that enlarges regions $P_{\mathrm{x}} := \{x \in \mathbb{R}^2 \,|\, p_{\mathrm{x}}(x) \leq c\}$ and $P_{\mathrm{z}} := \{z \in \mathbb{R}^2 \,|\, p_{\mathrm{z}}(z) \leq c\}$. We choose to use the same p_{x} as in the abovementioned state feedback examples, i.e. $p_{\mathrm{x}}(x) = \frac{1}{6}x_1^2 + \frac{1}{6}x_1x_2 + \frac{1}{12}x_2^2$, and $p_{\mathrm{z}}(z)$ is chosen to be of the same shape, i.e. $p_{\mathrm{z}}(z) := p_{\mathrm{x}}(z)$.

Even though we can directly measure x_2 in (5.62), optimization problem 5.4 searches for a full-state weak detector, so we will use the estimated states z_1 and z_2, instead of z_1 and x_2 for feedback.

We choose to search over homogeneous quadratic $\tilde{V}(x-z)$, $W_6(x-z)$, $s_1(x,z)$ and $s_2(x,z)$, and search over $\tilde{L}$ with linear and cubic terms only. The optimization returns a weak detector with $c = 0.1667$, which seems rather small compared to these provable regions of attraction when using state feedback: $\beta = 54.65$ in [16] and $\beta = 100$ in Section 5.2.2. However, we should view $c = 0.1667$ as a measure of the estimation error $x - z$: as long as the solution trajectories $x(t)$ and $z(t)$ do not leave P_{x} and P_{z} respectively, then $x(t) - z(t) \to 0$ as $t \to \infty$.

We perform some closed-loop simulations with $z(0) = [0; 0]$ and $x(0) = [-1; 2]$, which is just inside the level set P_{x} with $c = 0.1667$. We use controllers from the following two cases:

1. From [16], the state feedback controller is $K(x) = -145.94x_1 + 12.25x_2$, so we use $K(z) = -145.94z_1 + 12.25z_2$.
2. The controller constructed from the CLF in Section 5.2.2, where x in the formula (5.5) is replaced by z.

For case 1, Figure 5.5, shows that $x(t)$ and $z(t)$ do not leave P_{x} and P_{z} respectively,

Table 5.1. CDC'03 example: Closed loop system

q	degree of V	s_{6i}	s_{8i}	s_{9i}	s_{0ij}	β	total no. of decision variables
1	2	0	2	0	-	0.339	48
1	4	2	2	0	-	0.400	581
2	2	0	2	0	2	0.463	160
2	4	2	2	0	2	0.514	1182

so the estimation error indeed converges to zero and the origin is asymptotically stable by Theorem 5.1. For case 2, $x(t)$ does briefly leave P_{x} twice, as seen in Figure 5.6, but the estimation error still converges to zero.

Even though Theorem 5.1 states that the origin of the closed loop system is asymptotically stable using the estimated states as feedback, it does not quantify the system's region of attraction. We can use optimization problems 3.1 and 3.2 to quantify a provable region of attraction for case 1 as the vector field is polynomial. However we cannot directly do the same for case 2 as the controller is non-polynomial.

We choose $p(x,z) = p_{\mathrm{x}}(x)+p_{\mathrm{x}}(z)$ as the ellipsoid to enlarge. Table 5.1 shows the results for case 1, with the set $\{(x,z)\,|\,p(x,z) \le \beta\}$ quantifying the size of a provable region of attraction $\{(x,z)\,|\,V(x,z) \le 1\}$. We will use the results from the pointwise maximum of two quartic V's to illustrate the extent of its provable region of attraction. At $z(0) = [0;0]$, the region enclosed by the diamond-shaped curve is the set $\{x\,|\,\max\{V_1(x,0), V_2(x,0)\} \le 1\}$, as shown in Figure 5.7. Four initial conditions for $x(0)$ are picked so that they are near the boundary of this set. Simulation results with these initial conditions showed that all the trajectories starting from these initial conditions converge to the origin. Note that as $x(t)$ and $z(t)$ evolve, the $x_1 - x_2$ sectional view of provable region of attraction $\{(x,z)\,|\,\max\{V_1(x,z), V_2(x,z)\} \le 1\}$ changes, so it is not the diamond-shaped curve anymore, but the trajectories still remain within this region.

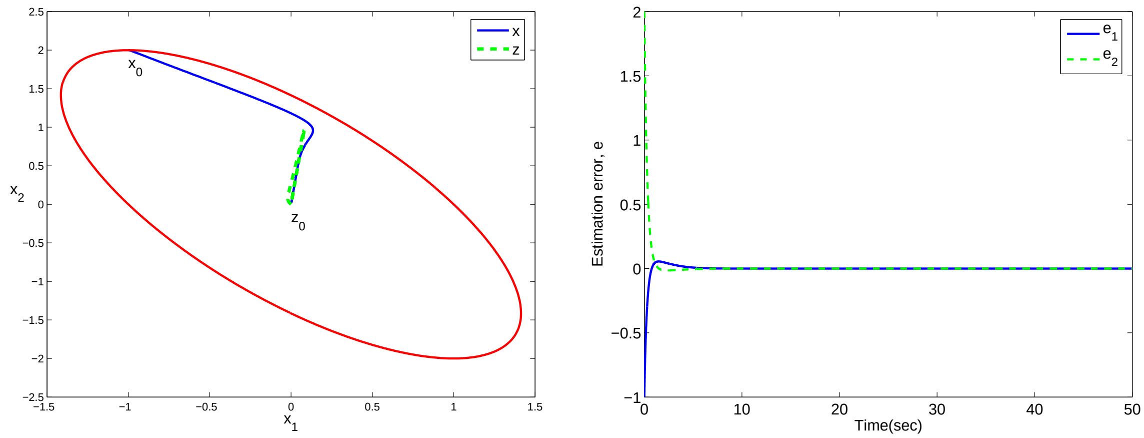

Figure 5.5. CDC'03 example (Case 1): Left: Estimate vs actual states; Right: Estimation errors

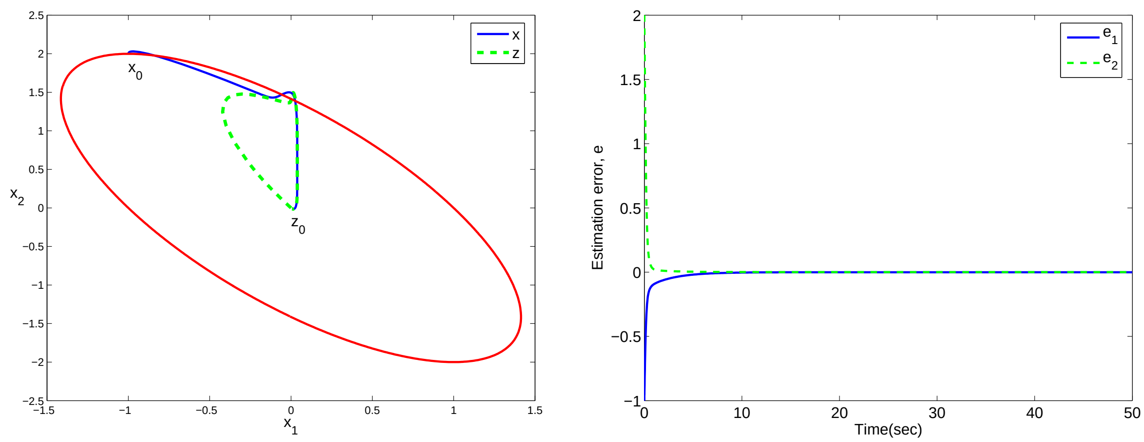

Figure 5.6. CDC'03 example (Case 2): Left: Estimate vs actual states; Right: Estimation errors

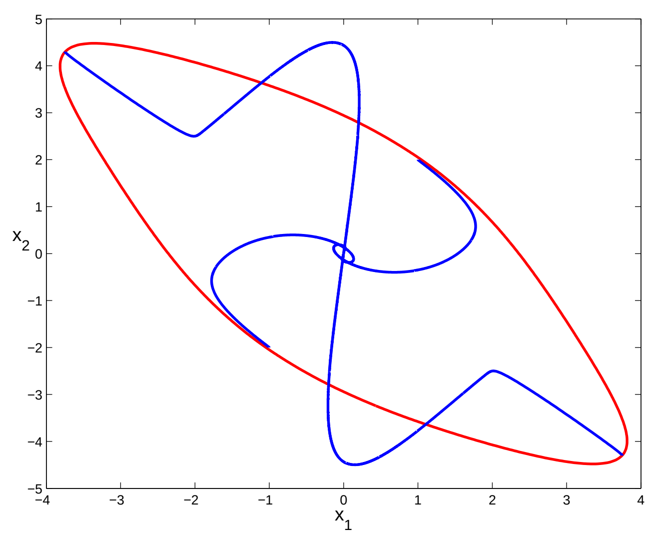

Figure 5.7. CDC'03 example using estimated states for feedback with larger x_0

5.4 Chapter Summary

In the first part of this chapter, we formulated the search for global and local CLFs for polynomial systems that are affine in control as SOS programming problems. We demonstrate our techniques with some non-trivial examples. Traditionally, construction of CLFs requires intimate knowledge of the system involved before one can even propose a likely CLF candidate. Moreover, finding a CLF is often by trial-and-error. We hope that with our method, finding CLFs for this class of nonlinear systems will be simplified and made systematic.

In the second part of this chapter, we formulated the search for nonlinear observers for polynomial systems as SOS programming problems, using Lyapunov based methods. The method proposed is rudimentary as it does not address the transient behavior of the observer, or its sensitivity to measurement noise and model uncertainties. One possibility is to combine the robustness analysis in Section 3.3 and the induced $\mathcal{L}_2$ to $\mathcal{L}_2$ gain of measurement noise to estimation error in Section 4.2 into our observer design. However, this combination would be computationally expensive due to additional variables needed to describe model uncertainties and measurement noises.

Chapter 6

Conclusions and Recommendations

This thesis considered Lyapunov based control analysis and synthesis methods for continuous time nonlinear systems with polynomial vector fields. We take the optimization approach of finding the Lyapunov functions through the use of SOS programming and the application of the Positivstellensatz theorem.

In chapter 3, we presented SOS programs that enlarge a provable region of attraction for polynomial systems. We proposed using pointwise maximum and minimum of Lyapunov functions to reduce the number of decision variables and to obtain larger inner bounds on the region of attraction. This idea is illustrated most notably with the Van der Pol equations example. We also extended this region of attraction enlargement problem to polynomial systems with uncertain dynamics by considering both parameter-dependent and independent Lyapunov functions. Besides using the pointwise maximum of such functions, we also proposed gridding the uncertain parameter space to further reduce the size of the SOS program. The significance of the gridding method is made apparent with two examples. A related stability region analysis problem of finding a tight outer bound for attractive invariant sets is also studied. Finally, we presented some computation statistics on a region of attraction benchmark example with arbitrary data and increasing problem size.

In chapter 4, we studied two local performance analysis problems for polynomial systems. The first is on finding upper bounds for the reachable set subjected to disturbances with $\mathcal{L}_2$ and $\mathcal{L}_\infty$ bounds. A SOS based refinement of the upper bound is proposed and illustrated with a previously studied example. The second problem is on finding an upper bound for the $\mathcal{L}_2 \rightarrow \mathcal{L}_2$ gain and its refinement. Interesting results are obtained when this method is applied to an adaptive control example.

In chapter 5, we studied controller and observer synthesis for polynomial systems. For controller synthesis, we presented SOS programs for finding global and local Control Lyapunov Functions. For observer synthesis, we formulated SOS programs that search for polynomial observers using Lyapunov based methods. Examples are provided to demonstrate the synthesis methods in this chapter.

It is hoped that the optimization based methods in this thesis will complement existing nonlinear analysis and design methods. There are several research directions that one can take from here:

1. In chapters 3, 4 and 5, we consider robust stability, performance and synthesis separately. In order for a control methodology to be truly useful in any practical settings, what is needed is to have robust performance specifications built into synthesis problems. Take for example, the nonlinear observer synthesis as presented in Chapter 5. The method proposed is rudimentary as it does not address the transient behavior of the observer, or its sensitivity to measurement noise and model uncertainties.

One possibility is to combine the robustness analysis in Section 3.3 and the induced $\mathcal{L}_2$ to $\mathcal{L}_2$ gain of measurement noise to estimation error in Section 4.2 into our observer design. However, this combination would be computationally expensive due to additional variables to describe model uncertainties and measurement noises.

2. As the SOS program problem size increases exponentially with the increase in the number of variables and the degree of the polynomial, this lack of scalability will ultimately limit the size of the problems considered. One approach is to break a complicated system into smaller nonlinear systems and consider the input-to-output behaviors of these smaller nonlinear systems and try to draw conclusions about the behavior of the interconnections of such system. To this end, the nonlinear small gain theorem and the integral quadratic constraint (IQC) methodology appear to show some promise.

3. The other direction to tackle this non-scalability of SOS programming is to exploit the structure and sparsity of the problem so that the SOS programs can be formulated into SDPs that are of smaller problem sizes. Some headway were made in [26] in this aspect and it would be worthwhile to continue research into this area.

4. Our local analysis and synthesis formulation resulted in SOS programs that are bilinear in the decision polynomials. This necessitates an efficient, global bilinear solver whose role is partially filled by PENBMI, a local bilinear solver. It is hoped that there will be significant advances in the development of efficient global bilinear solvers in the future.

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Appendix A

Practical Aspects of using SOS Programming

This appendix is written to help the reader with some practical aspects of using SOS programming tools, based on my anecdotal experience.

A.1 Constraints on the Degrees of Polynomials

A.1.1 Low degree terms in SOS multipliers

Since a Lyapunov function is required to be positive definite on a domain containing the origin, a polynomial Lyapunov function $V(x)$ must have $V(0) = 0$ and $V(x)$ must not contain any linear terms. As a result, $V(x)$ has quadratic and/or higher terms only.

The lack of a constant term in $V(x)$ imposes some constraints on the selection of monomials in the SOS multipliers. Take for example constraint (3.16) in Section 3.1.1:

$$-s_8 + V s_8 - \tfrac{\partial V}{\partial x} f s_9 - l_2 \in \Sigma_n \tag{A.1}$$

Since $f(0) = 0$, $\frac{\partial V}{\partial x} f s_9$ does not contain any constant or linear terms. Moreover, l_2 is a positive definite function, so $l_2(0) = 0$. If we allow the search of $s_8(x)$ with constant terms, $V s_8$ will have quadratic terms and above. Hence (A.1) will only have constant term due to $-s_8$, so $-s_8(0) \geq 0$. As $s_8 \in \Sigma_n$, $s_8(0) \geq 0$; this condition together with (A.1) imposes $s_8(0) = 0$, i.e. no constant term. In practice, a numerical solver cannot set this constant term to a hard zero, but will push it to a very small positive number. As a result, by allowing s_8 to have a constant term often leads to numerical problems. With s_8 chosen to have quadratic and/or higher terms, (A.1) will only have quadratic and higher terms as well, leading to less decision variables in the affine subspace. Summarizing, for SOS conditions that require $\frac{\partial V}{\partial x} f < 0$ on the set $\{x \,|\, V \leq 1\}$, the SOS multiplier associated with the term $(1 - V)$ should not have a constant term.

A.1.2 Using local analysis for global problems

On a related note, it is not recommended to use a local version of nonlinear analysis when global results are expected. For example, suppose a polynomial system is globally asymptotically stable and there exists a positive definite and radially unbounded V such that $-\frac{\partial V}{\partial x} f s_9 - l_2 \in \Sigma_n$. If, however, we use (A.1) instead to certify global stability, setting $s_8 \equiv 0$ should recover the global asymptotic stability condition. However, in practice, $s_8 \equiv 0$ cannot be obtained by the solvers, and may give infeasible results or numerical problems as the entries of V are scaled smaller and smaller so that $\{x \,|\, V(x) \leq 1\}$ represents an increasingly larger region. The message here is not to use local analysis when global analysis is needed.

A.1.3 Overall degree of SOS constraints

It is also useful to examine the degree of each term in a SOS constraint. Take for example the simple SOS condition $V - l_1 \in \Sigma_n$, where l_1 is a user chosen positive definite polynomial. If V is chosen to be quartic and l_1 is chosen to be of degree 6, $V - l_1 \in \Sigma_n$ cannot be satisfied because the highest degree terms are from $-l_1$, and their coefficients are negative. Choosing degrees of V and l_1 such that $\deg(V) \geq \deg(l_1)$ will prevent infeasibility due to such degree constraints.

For some other SOS constraints, it might not be so obvious as the previous example. Take for example in the CLF formulation, constraint (5.16) in Section 5.1.2:

$$- \left\{ s_1 \left[\tfrac{\partial V}{\partial x} f(x) \right] \; + \; p_2 \left[\tfrac{\partial V}{\partial x} g(x) \right] \; + \; l_2 \right\} \; \in \; \Sigma_n$$

Since $p_2, g \in \mathcal{R}_n$, the term $p_2 \left[\tfrac{\partial V}{\partial x} g \right]$ is not sign definite, so it is not clear whether this term should have degree greater or less than the other terms. One particular choice that I have used is to set the degrees of V, s_1, and p_2 such that both $s_1 \left[\tfrac{\partial V}{\partial x} f(x) \right]$ and $p_2 \left[\tfrac{\partial V}{\partial x} g(x) \right]$ have the same degree.

I have also used a similar idea for other SOS constraints, where the degree of each term except l_2 is set to be equal, whenever possible. Take for example, constraints (3.25) and (3.26) in the Van der Pol example. From Table 3.2, we can see that I have chosen the degrees of the SOS multipliers such that the degrees of ps_{6i} and V_i are equal (p is quadratic). With the degree of s_{6i} chosen, there is no more leeway to choose the degree of βs_{6i} as β is a scalar. Also $V_i s_{8i}$, $\frac{\partial V_i}{\partial x} f s_{9i}$ and $s_{0ij}(V_i - V_j)$ are chosen to have the same degree.

A.1.4 Terms with small coefficients

After optimization, it might be insightful to examine the coefficients of the decision polynomials returned by the solver. When there are entries that are 5 or 6 orders of magnitude smaller than other entries, it might indicate that those small entries are not needed and including them in the optimization might lead to numerical errors. For example, in Section 5.3.3.1, where we are synthesizing a nonlinear observer L for the Duffing equations, the quadratic terms of L have very small values, which is not surprising since the Duffing equations have linear and cubic terms only. After the quadratic terms of L have been excluded, we re-run the optimization again and we still obtain feasible results, indicating that the excluded quadratic terms are redundant.

A.2 Reducing decision variables in the affine subspace

In using YALMIP to solve SOS programming problems, we often need to search over SOS multipliers. The following two forms of declaring a SOS multiplier s_6 may look similar, but there is a significant difference in the number of decision variables involved:

```
sdpvar x1 x2;
z6 = monolist([x1; x2],3);
s6M = sdpvar(size(z6,1));
s6 = z6'*s6M*z6;
F = set(sos(s6));
```

```
sdpvar x1 x2;
z6 = monolist([x1; x2],3);
s6M = sdpvar(size(z6,1));
s6 = z6'*s6M*z6;
F = set(s6M >= 0);
```

The code fragment on the LHS explicitly tells YALMIP that s_6 has to satisfy the condition $s_6 \in \Sigma_n$. The number of variables involved in the affine subspace of `s6M` when using the image representation (sos.model == 2) is 27 (from Table 2.1, $n = 2, 2d = 6$).

In contrast, the code fragment on the RHS tells YALMIP that s_6 is a SOS polynomial by imposing that its Gram matrix `s6M` is positive semidefinite. Since we are searching over the decision polynomial s_6, and recall that $s_6 \in \Sigma_n$ if and only if there exists a positive semidefinite `s6M`, the two forms of declarations are equivalent. However, the RHS code does not incur the additional 27 decision variables in the affine subspace.

Another place where we can reduce the number of decision variables in the affine subspace is the V positive definite constraint. Note that $V - l_1 \in \Sigma_n \Leftrightarrow \exists V_0(x) \in \Sigma_n, V_0(0) = 0$ such that $V(x) = V_0(x) + l_1(x)$. The following two code fragments are similar, but again, the LHS incurs the additional decision variables while the RHS does not.

```
sdpvar x1 x2;
l1 = 0.01*(x1^2 + x2^2);
z = monolist([x1; x2],3);
z = z(2:end); % Remove constant term
P = sdpvar(size(z,1));
V = z'*P*z - l1;
F = set(sos(V)) + set(P >= 0);
```

```
sdpvar x1 x2;
l1 = 0.01*(x1^2 + x2^2);
z = monolist([x1; x2],3);
z = z(2:end); % Remove constant term
P = sdpvar(size(z,1));
V0 = z'*P*z; V = V0 + l1;
F = set(P >= 0);
```

Recall the Van der Pol equations example in Section 3.1.4.1. Table A.1 shows the difference in the number of decision variables when using the two different forms of declaring V and the SOS multipliers, as illustrated in all of the code fragments above. The difference becomes significant when the degrees of the polynomials are high.

Table A.1. VDP (Single V): Comparison of no. of decision variables

degree of				total no. of decision variables	
V	s_6	s_8	s_9	LHS	RHS
2	0	2	0	13	13
4	2	2	0	60	57
6	4	2	0	192	166
8	6	2	0	482	392
10	8	2	0	1017	795

A.3 Using YALMIP and PENBMI

In this section, I will recommend some settings for YALMIP and PENBMI to avoid numerical problems. In addition, I will highlight an easy method to force PENBMI to search over different feasible regions in order to improve upon the local optimal results.

A.3.1 Bounding the objective function

Since PENBMI is a local BMI solver, convergence to the global minimum is not guaranteed. For each run of PENBMI, it starts with randomized initial conditions, so with repeated runs, we might get the global minimum. One easy way to help this process along is to impose upper and lower bounds on the objective function to restrict the feasible region that PENBMI is allowed to search over.

Take for example, in the Van der Pol equations example in Section 3.1.4.1, we are maximizing the objective function β, which is a measure of the provable region of attraction. Of particular use is the setting of an increasingly larger lower bound on β after each successful run. For example, in searching for a degree 6 single Lyapunov function, we obtain $\beta = 0.781$ after numerous repeated runs. Since we know that $\beta = 0.781$ is achievable, we can set the lower bound of β to 0.8 so that PENBMI will not converge to that local maximum, but will try to find another better local maximum. After several such increasing lower bounds, we are able to obtain $\beta = 0.909$.

A.3.2 Settings for YALMIP

By default, YALMIP automatically does scaling for SOS programs. As a first step, try turning off the scaling if you encounter numerical problems:

```
opts = sdpsettings;
opts.sos.scale = 0;
        ⋮
solvesos(F,-beta1,opts);
```

A.3.3 Settings for PENBMI

There are settings in PENBMI that can be useful in resolving numerical problems or infeasibility due to numerical problems. Oft-times, I noticed that as the PENBMI iteration progresses, the two columns that represent slackness in linear and matrix multipliers, `feas` and `<U,A(x)>`, show small positive values, indicating that the problem is already feasible, so the optimization is trying to minimize the objective function as much as possible. However, due to possible ill-conditioning of multiplier matrices update during the iterations, the values displayed in the `<U,A(x)>` column start to increase rapidly and eventually, we get infeasible results. In order to terminate the optimization earlier before such ill-conditioning occurs, the following settings for PENBMI have been useful:

```
opts.penbmi.PBM_MAX_ITER = 500;
opts.penbmi.PBM_EPS = 2.5e-6;
opts.penbmi.PRECISION_2 = 1e-6;
opts.penbmi.MU2 = 0.05;
```

The stopping criteria [13] for PENBMI are 1) when the normalized difference between the objective function and the augmented Lagrangian is less than PBM_EPS, and 2) the rate of change in the objective function value between iterations is less than PBM_EPS. Since PBM_EPS is the tolerance for these two stopping criteria, relaxing this tolerance results in the optimization stopping earlier. PRECISION_2 is the tolerance of the KKT optimality conditions. MU2 is the update step size for the matrix multipliers. Setting this to a smaller value helps to prevent ill-conditioning of the matrix multipliers during update. Since the step size for the matrix multipliers is reduced, it might take more iterations to reach the optimal solution, so setting PBM_MAX_ITER to a larger number prevents premature termination due to maximum number of iterations being reached.

A.3.4 Scaling the objective function

Since PBM_EPS is the tolerance for two stopping criteria, we cannot independently change one of the criteria with PBM_EPS. However, we can do so by scaling the objective function. Recall that one of the stopping criteria is that the rate of change in the objective function value between iterations is less than PBM_EPS. By scaling the objective function to a smaller value, say $\beta \to 0.01 * \beta$, the optimization will terminate earlier, at the expense of obtaining a sub-optimal solution.