Matrix Representations of Polynomials: Theory and Applications

by

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Abstract

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This report considers a nonlinear state transformation that possesses certain properties that make it amenable to controls problems. With this state transformation, we can form matrix representations of high degree multivariable polynomials, which allows us to use techniques of linear algebra and quadratic polynomials to gain a greater understanding of these higher degree polynomials.

Representing polynomials as matrices gives us an LMI test to see if a polynomial is a sum of squares polynomial as well as providing a generalization of the $\mathcal{S}$-procedure to include higher degree polynomials along with the standard quadratic forms. The representation of a higher degree polynomial under the nonlinear transform also gives a method for fitting data to both sum of squares and general polynomials

Lastly considered in this report are sum of squares polynomial Lyapunov functions to demonstrate simultaneous stability for a finite collection of linear systems. We show that the minimum degree sum of squares Lyapunov function that demonstrates simultaneous stability for a collection of linear systems can be written as a homogeneous polynomial, subject to a weak definiteness condition. The resulting approach to finding a sum of squares Lyapunov function allows us to improve the performance of a bench mark example problem, as well as consider the stability of an observer that receives intermittent information from the plant.

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Chapter 1

Introduction

Studying new problems by reformulating them into a more familiar framework allows you to use your existing expertise in one area to gain a deeper understanding of new one; in control engineering, linear algebra is a well understood tool into whose forms many problems can be recast. This report is based around the idea of using a nonlinear state transform to study and exploit the characteristics of multivariable high degree polynomials, which are not standard control engineering tools, with linear algebra.

By working with the polynomials in the familiar forms of vectors and matrices, we can expand results relating to quadratic polynomials to ones of arbitrary degree. The familiarity that we gain with high degree polynomials under the nonlinear state transformation, does come at the price of unique representations.

Even with the troubles of non-uniqueness, this transformation allows us to to form the central result of Chapter 4, that limits the size of the search for a positive definite function, as an LMI feasibility problem. The convexity of such a search is a very pleasing byproduct of using the transform, and the problem formulation allows us to easily consider a large set of examples.

Chapter 2

Power State Transformation

2.1 Definition of the transform

Definition 2.1 *Consider the vector* $x \in \mathbb{R}^n$*,*

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

the ***power transformation of degree*** p *is a nonlinear change of coordinates that forms a new vector* $x^{[p]}$ *of all integer powered monomials of degree* p *that can be made from the original* x *vector,*

$$x_l^{[p]} := c_l x_1^{p_{l1}} x_2^{p_{l2}} \cdots x_n^{p_{ln}} \qquad l = 1, \cdots, m$$

with $c_l \in \mathbb{R}$*,* $m = \binom{n+p-1}{p}$*,* $p_{lj} \in \{0, 1, \ldots, p\}$*, the restriction* $\sum_j p_{lj} = p, \forall l$*, and where the ordering of the* $x_l^{[p]}$*'s is lexigraphic in the* p_{lj}*'s.*

Note that $x^{[0]} = c_1$, and $x^{[1]} = Cx$, with $C = \text{diag}(c_1, \cdots, c_m)$. Also, since all the $x_l^{[p]}$'s have different combinations of powers of the x_i's, they are linearly independent, but it is important to remember that the $x_l^{[p]}$'s are not independent.

2.1.1 Length preservation under the transform

Usually we will take the $c_l = 1$, $\forall l$, to simplify notation and calculation, but if we let

$$c_l = \sqrt{\binom{p}{p_{l1}}\binom{p - p_{l1}}{p_{l2}} \cdots \binom{p - p_{l1} - \cdots - p_{l(n-1)}}{p_{ln}}}$$

or equivalently

$$c_l = \sqrt{\frac{p!}{p_{l1}!p_{l2}! \cdots p_{ln}!}}$$

we have the following lemma originating in [Bro73].

Lemma 2.1 *Letting $< x, y >$ be the ordinary Euclidean inner product, the following relation holds*

$$< x, y >^p = < x^{[p]}, y^{[p]} >$$

Proof:

Working out the details of the inner product gives the desired result.

$$\begin{aligned}
< x, y >^p &= \left[\sum_{i=1}^{n} x_i y_i\right]^p \\
&\overset{(a)}{=} \sum_{l=1}^{m} \frac{p!}{p_{l1}!p_{l2}! \cdots p_{ln}!}[x_1 y_1]^{p_{l1}}[x_2 y_2]^{p_{l2}} \cdots [x_n y_n]^{p_{ln}} \\
&= \sum_{l=1}^{m} \left[\sqrt{\frac{p!}{p_{l1}!p_{l2}! \cdots p_{ln}!}} x_1^{p_{l1}} \cdots x_n^{p_{ln}}\right] \left[\sqrt{\frac{p!}{p_{l1}!p_{l2}! \cdots p_{ln}!}} y_1^{p_{l1}} \cdots y_n^{p_{ln}}\right] \\
&= \sum_{l=1}^{m} x_l^{[p]} y_l^{[p]} \\
&= < x^{[p]}, y^{[p]} >
\end{aligned}$$

where equality (a) is from the multinomial formula.

□

Corollary 2.1

$$\|x\|_2^p = \|x^{[p]}\|_2$$

Note: From here on, we will assume that all the $c_l = 1$.

2.1.2 Examples of the transform

The following are examples of the power transform:

1. The most basic form is $n = p = 2 \quad \Rightarrow \quad m = 3$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow \quad x^{[2]} = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

2. $n = 2$, $p = 4 \quad \Rightarrow \quad m = 5$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow \quad x^{[4]} = \begin{bmatrix} x_1^4 \\ x_1^3 x_2 \\ x_1^2 x_2^2 \\ x_1 x_2^3 \\ x_2^4 \end{bmatrix}$$

3. $n = 3$, $p = 3 \quad \Rightarrow \quad m = 10$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \Rightarrow \quad x^{[3]} = \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1^2 x_3 \\ x_1 x_2^2 \\ x_1 x_2 x_3 \\ x_1 x_3^2 \\ x_2^3 \\ x_2^2 x_3 \\ x_2 x_3^2 \\ x_3^3 \end{bmatrix}$$

2.2 Relations under the transform

The power transformation preserves certain linear relations which makes the transform useful for studying linear systems.

2.2.1 Definition and properties of $A^{[p]}$

Lemma 2.2 *Given $A \in \mathbb{R}^{r\times n}$ and $p \in \mathbb{Z}_+$, then under the power transform, $\exists A^{[p]} \in \mathbb{R}^{\binom{r+p-1}{p}\times\binom{n+p-1}{p}}$ such that $(Ax)^{[p]} = A^{[p]}x^{[p]}$, $\forall x \in \mathbb{R}^n$.*

Proof:

Let $y = Ax$, and by induction with the original linear equation

$$y^{[1]} = y = Ax = Ax^{[1]} \Rightarrow A^{[1]} = A$$

as the base case, we will assume that there exists an $A^{[n]}$ such that $y^{[n]} = A^{[n]}x^{[n]}$ holds. Each $y_l^{[n+1]} = y_j^{[n]}y_k$ for some j, k. Since $y_j^{[n]}$ is a linear combination of monomials in the x_i's of degree n and y_k is a linear combination of the x_i's, their product will just be a linear combination of monomials in the x_i's of degree $n+1$. This implies that $y_l^{[n+1]}$ is a linear combination of degree $n+1$ monomials in the x_i's giving the lth row of $A^{[n+1]}$, so $y^{[n+1]} = A^{[n+1]}x^{[n+1]}$.

□

We know more about $A^{[p]}$ and its structure from its construction.

Lemma 2.3 *The elements of $A^{[p]}$ are homogeneous polynomials of degree p in the elements of A. Definition 3.2 gives a full definition of homogeneous polynomials.*

Proof:

Given $y = Ax$, each y_i is linear in the elements of A, and each $y_l^{[p]}$ is a monomial of degree p in the y_i's, which forces these products to be homogeneous polynomials of degree p in the elements of A.

□

Theorem 2.1 *Given A, $B \in \mathbb{R}^{n\times n}$, $A^{[p]}$ and $B^{[p]}$ satisfy*

1. $(AB)^{[p]} = A^{[p]}B^{[p]}$
2. $(A^q)^{[p]} = (A^{[p]})^q$*, with q integer and A^q well defined.*

3. $(A^*)^{[p]} = (A^{[p]})^*$

(This is Theorem 1 in [Bro73])

Proof:

1. Let $z = Ay = ABx$, then $z^{[p]} = A^{[p]}y^{[p]} = A^{[p]}B^{[p]}x^{[p]} = (AB)^{[p]}x^{[p]}$.
2. Use (1.) with $B = A$ (or $B = A^{-1}$) and go by induction.
3. Using Lemmas 2.1 and 2.2,

$$\begin{aligned} < x^{[p]}, (A^{[p]})^* x^{[p]} > &= < A^{[p]}x^{[p]}, x^{[p]} > \\ &= < Ax, x >^p \\ &= < x, A^*x >^p \\ &= < x^{[p]}, (A^*)^{[p]} x^{[p]} > \end{aligned}$$

Since these equalities hold for any x, $(A^*)^{[p]} = (A^{[p]})^*$.

□

Examples

1. For $r = n = p = 2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \Rightarrow \quad A^{[2]} = \begin{bmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{bmatrix}$$

2. For $r = 1$, $n = 2$, $p = 4$

$$A = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \quad \Rightarrow \quad A^{[4]} = \begin{bmatrix} a_{11}^4 & 4a_{11}^3 a_{12} & 6a_{11}^2 a_{12}^2 & 4a_{11}a_{12}^3 & a_{12}^4 \end{bmatrix}$$

3. For $r = 2$, $n = 3$, $p = 2$

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\ &\Downarrow \\ A^{[2]} &= \begin{bmatrix} a_{11}^2 & 2a_{11}a_{12} & 2a_{11}a_{13} & a_{12}^2 & 2a_{12}a_{13} & a_{13}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{11}a_{23} + a_{13}a_{21} & a_{12}a_{22} & a_{12}a_{23} + a_{13}a_{22} & a_{13}a_{23} \\ a_{21}^2 & 2a_{21}a_{22} & 2a_{21}a_{23} & a_{22}^2 & 2a_{22}a_{23} & a_{23}^2 \end{bmatrix} \end{aligned}$$

2.2.2 Definition and properties of $A_{[p]}$

Theorem 2.2 *Given $A(t) \in \mathbb{R}^{n\times n}$ and $p \in \mathbb{Z}_+$, with the relation $\dot{x}(t) = A(t)x(t)$, then there exists a matrix $A_{[p]}(t) \in \mathbb{R}^{\binom{n+p-1}{p}\times\binom{n+p-1}{p}}$ such that $\frac{d}{dt}\left(x^{[p]}(t)\right) = A_{[p]}(t)x^{[p]}(t)$, and the operation that maps A into $A_{[p]}$ is linear.*

Proof:

Looking to the original differential equation and the $\frac{d}{dt}(x_l^{[p]})$'s, we see that by the product rule each $\frac{d}{dt}(x_l^{[p]})$ will be linear in the $\dot{x}_i$'s. Each $\dot{x}_i$ is a linear combination of the A_{ij}'s, and this combination of linearity means that the resulting products will remain linear in A_{ij}'s. These linear relations construct an $A_{[p]}$ such that its elements, the $A_{[p]ij}$'s, are linear combinations of the A_{ij}'s, showing that the map from A to $A_{[p]}$ is linear.

□

Alternatively we could follow [Bro73] and construct $A_{[p]}$ through the definition of the time derivative of $x^{[p]}$. Either way we get the very important corollary.

Corollary 2.2 *Letting α, $\beta \in \mathbb{R}$ and $A, B \in \mathbb{R}^{n\times n}$,*

$$[\alpha A + \beta B]_{[p]} = \alpha A_{[p]} + \beta B_{[p]}$$

Eigenvalues of $A_{[p]}$

Lemma 2.4 *If A is Hurwitz, then $A_{[p]}$ is as well.*

Proof:

Stability of $A_{[p]}$ can be demonstrated for any diagonalizable A matrix by taking the power transform of its diagonalization, TAT^{-1}, which with Theorem 2.1 becomes $T^{[p]}A_{[p]}(T^{[p]})^{-1}$. Since TAT^{-1} was diagonal we have equations of the form

$$\dot{x}_i = \lambda_i x_i$$

with the λ_i's the eigenvalues of A. Forming the power transform products for fixed p and taking time derivatives yields

$$\frac{d}{dt}(x_l^{[p]}) = \frac{d}{dt}\left(x_1^{p_{l1}} \dots x_n^{p_{ln}}\right) = \left(p_{l1}\lambda_1 + \dots + p_{ln}\lambda_n\right)\left(x_1^{p_{l1}} \dots x_n^{p_{ln}}\right)$$

for each $l = 1, \ldots, m$, showing that $T^{[p]}A_{[p]}(T^{[p]})^{-1}$ is indeed diagonal. Also, by the definition of the transform, each of the p_{li}'s is non-negative, they all sum to p, and, by assumption, each λ_i has negative real part. Since $T^{[p]}A_{[p]}(T^{[p]})^{-1}$ is diagonal, this combination of the eigenvalues of A implies that $T^{[p]}A_{[p]}(T^{[p]})^{-1}$'s eigenvalues are in the left half plane and their real part is bounded away from zero, which in turn implies that $A_{[p]}$ is Hurwitz.

If A is not diagonalizable we can argue stability by continuity. Let $\gamma = |\max_i \mathbf{Re}(\lambda_i)|$ and then form the Jordan form of A, $J = TAT^{-1}$. We can form a sequence of perturbations to A, $T^{-1}D_kT$, with D_k chosen such that $\bar{\sigma}(D_k) < \frac{1}{2}\gamma/k$ and $J_{ii} + (D_k)_{ii} \neq J_{jj} + (D_k)_{jj}$. These perturbations allow us to form the sequence of matrices

$$A_k = A + T^{-1}D_kT$$

which clearly converges to A. By our choice of D_k's, all of the A_k's are diagonalizable and stable, and from our results for diagonalizable matrices this implies that each of the $(A_k)_{[p]}$'s is stable with its eigenvalues' real parts bounded away from zero. By the continuity of the power transform, the $(A_k)_{[p]}$'s converge to $A_{[p]}$ and the bounds on the $(A_k)_{[p]}$'s eigenvalues keep the eigenvalues of $A_{[p]}$ entirely inside the left half plane, so $A_{[p]}$ is Hurwitz.

□

Examples

1. For $n = p = 2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \Rightarrow \quad A_{[2]} = \begin{bmatrix} 2a_{11} & 2a_{12} & 0 \\ a_{21} & a_{11} + a_{22} & a_{12} \\ 0 & 2a_{21} & 2a_{22} \end{bmatrix}$$

2. For $n = 2$, $p = 4$

$$
\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\
&\Downarrow \\
A_{[4]} &= \begin{bmatrix} 4a_{11} & 4a_{12} & 0 & 0 & 0 \\ a_{21} & 3a_{11} + a_{22} & 3a_{12} & 0 & 0 \\ 0 & 2a_{21} & 2a_{11} + 2a_{22} & 2a_{12} & 0 \\ 0 & 0 & 3a_{21} & a_{11} + 3a_{22} & a_{12} \\ 0 & 0 & 0 & 4a_{21} & 4a_{22} \end{bmatrix}
\end{aligned}
$$

3. For $n = 3$, $p = 2$

$$
\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
&\Downarrow \\
A_{[2]} &= \begin{bmatrix} 2a_{11} & 2a_{12} & 2a_{13} & 0 & 0 & 0 \\ a_{21} & a_{11} + a_{22} & a_{23} & a_{12} & a_{13} & 0 \\ a_{31} & a_{32} & a_{11} + a_{33} & 0 & a_{12} & a_{13} \\ 0 & 2a_{21} & 0 & 2a_{22} & 2a_{23} & 0 \\ 0 & a_{31} & a_{21} & a_{32} & a_{22} + a_{33} & a_{23} \\ 0 & 0 & 2a_{31} & 0 & 2a_{32} & 2a_{33} \end{bmatrix}
\end{aligned}
$$

Chapter 3

Matrix representation of polynomials using the power transform

The previous chapter provides us with the useful tools of the power transformation and allows us to work with high degree, high dimensional polynomials in the familiar setting of quadratic functions on matrices, but this advance does not come without its costs. If we use the power transform to represent polynomials as quadratic forms on matrices, we lose the uniqueness of the representation of the polynomials.

3.1 Definitions relating to polynomials

Before we can begin to study representations of polynomials under the power transformation, we need to make the following definitions about different classes of polynomials.

Definition 3.1 (Polynomials) *Let $P_{n,d}$ be the set of all polynomials in n variables with degree d.*

Definition 3.2 (Homogeneous Polynomials) *Let $H_{n,d}$ be the set of all homogeneous polynomials in n variables with degree d. $f \in H_{n,d}$ iff $f \in P_{n,d}$ and $f(\lambda x) =$*

$\lambda^d f(x)$.

The previous two definitions yield the trivial containment $H_{n,d} \subset P_{n,d}$. Also, we can see that any $f \in P_{n,d}$ is made up of sums of elements of $H_{n,i}$, $i = 0, \ldots, d$.

Definition 3.3 (Positive Semidefinite Polynomials) *Let $\Pi_{n,2d}$ be the set of all positive semidefinite polynomials in n variables with degree $2d$. $f \in \Pi_{n,2d}$ iff $f \in P_{n,2d}$ and $f(x) \geq 0$, $\forall x \in \mathbb{R}^n$.*

Definition 3.4 (Sum of Squares Polynomials) *Let $\Sigma_{n,2d}$ be the set of all sum of squares polynomials in n variables with degree $2d$. $f \in \Sigma_{n,2d}$ iff $\exists g_i \in P_{n,d}$ with $f = \sum_i (g_i)^2$.*

3.1.1 Some properties of the sets of polynomials

Working a few basic lemmas we get the building blocks of the later results.

Lemma 3.1 *$f \in P_{n,2t}$ if and only if $\exists M \in \mathbb{R}^{q \times q}$, with $q = \sum_{i=0}^{t} \binom{n+i-1}{i}$ such that $M = M^*$ and*

$$f = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix}^* \begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix}$$

for $x \in \mathbb{R}^n$.

Proof:

$\Leftarrow$ Working out the multiplication gives a polynomial in n variables of degree $2t$.

$\Rightarrow$ All the monomials of an $f \in P_{n,2t}$ can be factored into products of two monomials of degree less than or equal to t, so there is at least one, not necessarily symmetric, matrix R such that

$$f = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix}^* \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix}$$

Then take $M = \frac{1}{2}(R + R^*)$.

□

Remark: Lemma 3.1 is equivalent to the statement "All polynomials in x of degree $2t$ can be represented as at least one symmetric quadratic form in the vector $[1, x, \cdots, x^{[t]}]^*$."

This lemma works on odd degree polynomials as well by picking $2t$ to be greater than the odd degree. Looking at the set of homogeneous polynomials we get a similar result.

Lemma 3.2 *$f \in H_{n,2p}$ if and only if $\exists M \in \mathbb{R}^{\binom{n+p-1}{p} \times \binom{n+p-1}{p}}$ such that $M = M^*$ and $f = x^{[p]*} M x^{[p]}$ for $x \in \mathbb{R}^n$*

Proof:

$\Leftarrow$ We have $f = x^{[p]*} M x^{[p]}$, with $M = M^*$. By doing the multiplication we get $f \in P_{n,2p}$, and we have

$$f(\lambda x) = \left(\lambda^p x^{[p]}\right)^* M \left(\lambda^p x^{[p]}\right) = \lambda^{2p} x^{[p]*} M x^{[p]}$$

$\Rightarrow$ We know that f is a sum of degree $2p$ monomials, each of which can be factored into a product of 2 degree p monomials. Thus, this sum of products of 2 degree p monomials can be written as at least one quadratic form $f = x^{[p]*} N x^{[p]}$, with N not necessarily symmetric. We then let $M = \frac{1}{2}(N + N^*)$.

□

Remark: Lemma 3.2, is also equivalent to "All homogeneous polynomials in x of degree $2p$ can be represented by at least one symmetric quadratic form in $x^{[p]}$."

We can continue this string of lemmas establishing quadratic form representations of polynomials with one for sum of squares polynomials that comes from [PowW98].

Lemma 3.3 *$h \in \Sigma_{n,2t}$ if and only if $\exists S \in I\!R^{q\times q} \geq 0$, with $q = \sum_{i=0}^{t} \binom{n+i-1}{i}$ such that*

$$h = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix}^* \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix}$$

for $x \in I\!R^n$.

Proof:

$\Leftarrow$ Since $S \geq 0$, $\exists L \in \mathbb{R}^{k\times q}$ with k the rank of S, such that $S = L^*L$, which gives

$$h = \left(L \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix} \right)^* \left(L \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix} \right) = \sum_{i=0}^{k} \left(L_i \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix} \right)^2 = \sum_{i=0}^{k} (g_i)^2$$

with each $g_i \in P_{n,t}$, so $h \in \Sigma_{n,2t}$.

$\Rightarrow$ We have $h = \sum_{j=0}^{k} (g_i)^2$. Since each of the g_i's are in $P_{n,t}$, we can introduce l_i's $\in \mathbb{R}^q$, such that $g_i = [1, x, \cdots, x^{[t]}]^* l_i$. Letting $L^* = [l_1, \cdots, l_k]$, we get

$$h = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix}^* L^*L \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[t]} \end{bmatrix}$$

$L^*L \geq 0$, so we take $S = L^*L$.

□

Remark: Lemma 3.3 is equivalent to "All sum of squares polynomials in x of degree $2t$ can be represented by at least one positive semidefinite quadratic form in the vector $[1, x, \cdots, x^{[t]}]^*$."

3.1.2 Containment relations among the sets of polynomials

By Lemmas 3.1, 3.2, and 3.3, along with definitions of the various sets of polynomials, we get the containment result

$$\Sigma_{n,2t} \subset \Pi_{n,2t} \subset P_{n,2t}$$

The strict containment is clear except for whether there exists a positive polynomial that is not sum of squares, and an example demonstrating this strict containment is given later in the discussion of testing whether a polynomial is sum of squares.

3.1.3 Examples

1. $f = 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$

$$f = \begin{bmatrix} 1 \\ x \\ x^{[2]} \end{bmatrix}^* \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^{[2]} \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}^* \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}$$

Here $S \geq 0$ so $f \in \Sigma_{2,4}$, and also $f \in H_{2,4}$. Again, note that there are many possible S matrices, and not all of them will be positive semidefinite. The question of how to find a semidefinite one follows in §3.2. This example is adopted from [Par00].

2. $f = x^4 + 2x^3 + 2x^2 + 1$

$$f = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

since the matrix in the quadratic form is positive definite, $f \in \Sigma_{1,4}$.

3.2 Is a given polynomial sum of squares?

If given a polynomial, how do you tell if it belongs to the set Σ? At first glance it would seem that you need only find the matrix for the quadratic form, but since this matrix need not be unique, you might find a non-semidefinite one. This non-uniqueness has been known for quite some time, [BosL68], and was studied more in depth in [Par00].

By parameterizing the quadratic forms we can search over all possible representations. Defining

$$\hat{x} := \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[p]} \end{bmatrix}$$

we can write the polynomial as $f = \hat{x}^* Q \hat{x}$. To parameterize Q, we take Q_0 as a particular representation of the polynomial, such that $f = \hat{x}^* Q_0 \hat{x}$, and then we find all the symmetric Q_i such that $\hat{x}^* Q_i \hat{x} = 0$. These Q_i's can be thought of as some type of "homogeneous representations," to parallel the types of solutions of differential equations. With these matrices we can parameterize the equation as

$$f = \hat{x}^* \left[Q_0 + \sum_i \lambda_i Q_i \right] \hat{x}$$

Now we look to find if there exists at least one $Q \geq 0$, and, nicely, this is just the LMI feasibility problem

$$\begin{array}{ll} \exists? & \lambda_i \\ \text{s.t.} & Q_0 + \sum\limits_i \lambda_i Q_i \geq 0 \end{array} \tag{3.1}$$

The convexity of this feasibility problem gives us a definite answer to the question of whether or not the given polynomial has a sum of squares representation.

3.2.1 The Q_i's

Why should there be any other matrix aside from the zero matrix such that $\hat{x}^* Q_i \hat{x} = 0$? If we look back to the definition of the power transform, we see that the elements are linearly independent, but not independent. Considering the most basic case $n = p = 2$. We have

$$x^{[2]} = \begin{bmatrix} x_1^{[2]} \\ x_2^{[2]} \\ x_3^{[2]} \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

and $x_1^{[2]} x_3^{[2]} = \left(x_2^{[2]}\right)^2$. This relation gives the Q_1 matrix as

$$Q_1 = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$$

For this case of $n = p = 2$ this is the only Q_i matrix. In general for any p, n there will be $q_{n,p}$ of these matrices.

The relations that are described by the Q_i matrices are quadratic equalities that describe the cone on which the state vector, $x^{[p]}$, resides. These quadratic equalities are of the form

$$x_i^{[p]} x_j^{[p]} - x_k^{[p]} x_h^{[p]} = 0$$

which is easily translated in to a symmetric matrix by letting the (i, j) and (j, i) entries be 1 and the (k, h) and (h, k) entries be -1, or some scalar multiple thereof. Similar constructions can be done for the vector $\hat{x}$.

3.2.2 Example that fails the sum of squares test

This example shows that the set $\Sigma_{n,2d}$'s containment in the set $\Pi_{n,2d}$ is strict, and it is taken from §4.2 in [Par00].

Consider $f \in P_{3,6}$,

$$f = x^4 y^2 + x^2 y^4 + z^6 - 3x^2 y^2 z^2$$

In [Par00], it is argued that this example, which is a Mozkin form in three variables, can be shown to be positive semidefinite via the arithmetic-geometric inequality, so

$f \in \Pi_{3,6}$. To check if $f \in \Sigma_{3,6}$, we first find a particular representation as

$$f = \hat{x}^* Q_0 \hat{x} = \begin{bmatrix} x^3 \\ x^2y \\ x^2z \\ xy^2 \\ xyz \\ xz^2 \\ y^3 \\ y^2z \\ yz^2 \\ z^3 \end{bmatrix}^* \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^3 \\ x^2y \\ x^2z \\ xy^2 \\ xyz \\ xz^2 \\ y^3 \\ y^2z \\ yz^2 \\ z^3 \end{bmatrix}^*$$

Looking for all the quadratic combinations of states that yield equalities, we end up finding 36 Q_i's, so the existence of a sum of squares representation is the LMI feasibility problem of (3.1) in the 36 λ_i's.

The infeasibility of the LMI is given by a lower bound of $t = 0.178395$, from the dual, of the problem

$$\begin{array}{ll} \min & t \\ \text{s.t.} & Q_0 + \sum_{i=1}^{36} \lambda_i Q_i + tI > 0 \end{array}$$

This lower bound on t, shows that there exists no set of λ_i's such that the sum of squares form existence test will work, and thus that $f \in \Pi_{3,6}$ and $f \notin \Sigma_{3,6}$.

3.3 A generalized polynomial $\mathcal{S}$-Procedure

A frequent question in optimization is: "Does a series of constraints imply another constraint?" When the constraints are quadratic, we get the standard $\mathcal{S}$-procedure as given in [BoyEFB94]

Define the constraints

$$F_i(x) := \begin{bmatrix} 1 \\ x \end{bmatrix}^* M_i \begin{bmatrix} 1 \\ x \end{bmatrix}, \qquad i = 0, \ldots, k$$

with $M_i = M_i^*$. Consider the following containment relation on the constraints

$$\bigcap_{i=1}^{k} \{x : F_i(x) \geq 0\} \subseteq \{x : F_0(x) \geq 0\} \tag{3.2}$$

We would like to know if this containment relation holds, and a sufficient condition for (3.2) to hold is:

$$\begin{array}{rll} & \exists \tau_i \geq 0, & i = 1, \dots, k, \\ \text{s.t.} & F_0(x) - \sum_{i=1}^{k} \tau_i F_i(x) \geq 0 & \forall x \end{array}$$

or using the matrix representation of the quadratic constraints

$$\begin{array}{rll} & \exists \tau_i \geq 0, & i = 1, \dots, k \\ \text{s.t.} & M_0 - \sum_{i=1}^{k} \tau_i M_i \geq 0 & \end{array}$$

which is just an LMI, so we can check the sufficient condition with minimal computational effort. When $k = 1$ this condition is also necessary, but it is not trivial to show this.

We can now expand our set of constraints to be $P_{n,2p}$, and using Lemma 3.1 we can write the constraints as

$$G_i(x) := \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[p]} \end{bmatrix}^* \begin{bmatrix} & & \\ & N_i & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[p]} \end{bmatrix}, \qquad i = 0, \dots, k$$

with $N_i = N_i^*$.

Consider the containment condition parallel to (3.2)

$$\bigcap_{i=1}^{k} \{x : G_i(x) \geq 0\} \subseteq \{x : G_0(x) \geq 0\} \tag{3.3}$$

Again we would like to know if the containment above holds, and again a sufficient condition for (3.3) to hold is:

$$\begin{array}{rll} & \exists \tau_i \geq 0, & i = 1, \dots, k, \\ \text{s.t.} & G_0(x) - \sum_{i=1}^{k} \tau_i G_i(x) \geq 0 & \forall x \end{array}$$

substituting in the definitions of the G_i's

$$
\begin{array}{ll}
& \exists \tau_i \geq 0 \qquad\qquad\qquad i = 1, \ldots, k, \\
\text{s.t.} & \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[p]} \end{bmatrix}^* \left[N_0 - \sum_{i=1}^{k} \tau_i N_i \right] \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[p]} \end{bmatrix} \geq 0 \quad \forall x
\end{array}
$$

If we can make the matrix in the quadratic form above positive semidefinite then we have a sufficient condition for (3.3). Remembering about the Q_i's, such that $\hat{x}^* Q_i \hat{x} = 0$, we get

$$
\begin{array}{ll}
& \exists\, \tau_i \geq 0, \quad \lambda_j \qquad\qquad\qquad i = 1, \ldots, k, \quad j = 1, \ldots, h \\
\text{s.t.} & N_0 - \sum_{i=1}^{k} \tau_i N_i + \sum_{j=1}^{h} \lambda_j Q_j \geq 0
\end{array}
\tag{3.4}
$$

which is, again, just an LMI. Note also that if we allowed the τ_i's to be positive semidefinite functions of x the same sufficient condition would hold. If the τ_i's were polynomials, in addition to being positive semidefinite functions of x, then we could form the matrix representation of the product $\tau_i(x) G_i(x)$ and substitute these matrices for the N_i's in (3.4).

3.3.1 Example of a polynomial $\mathcal{S}$-procedure

Letting $G_0(x) = x^4 - \alpha_0 x^2$, and $G_1(x) = x^4 - \alpha_1 x^2$, allows us to form a sufficient condition for (3.3) as

$$
\begin{array}{ll}
& \exists \tau_1 \geq 0 \\
\text{s.t.} & \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^* \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \tau_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \geq 0 \quad \forall x
\end{array}
$$

which can be set in the form of the LMI feasibility problem of (3.4) as

$$
\begin{array}{ll}
& \exists \tau_1 \geq 0, \quad \lambda_1 \\
\text{s.t.} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \tau_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \geq 0
\end{array}
$$

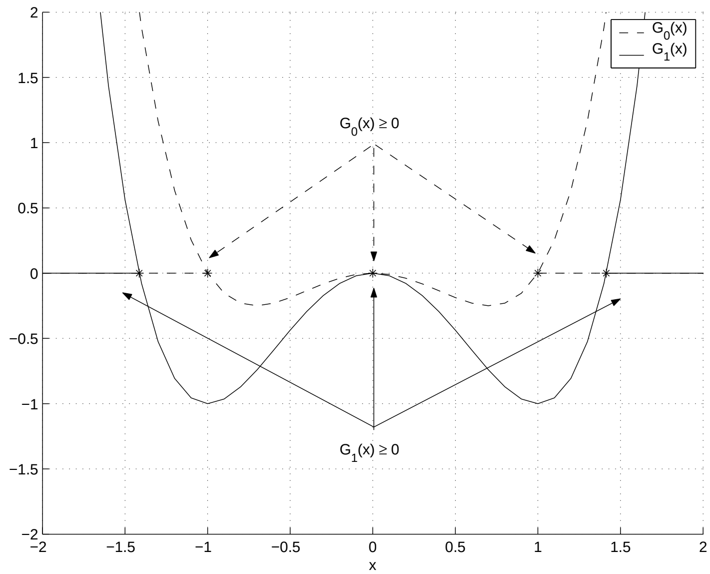

Figure 3.1: Polynomial $\mathcal{S}$-procedure for $G_0(x) = x^4 - x^2$ and $G_1(x) = x^4 - 2x^2$

or more succinctly

$$\exists \tau_1 \geq 0, \quad \lambda_1$$
$$\text{s.t.} \quad \begin{bmatrix} 0 & 0 & \lambda_1 \\ 0 & \tau_1\alpha_1 - \alpha_0 - 2\lambda_1 & 0 \\ \lambda_1 & 0 & 1 - \tau_1 \end{bmatrix} \geq 0$$

Via the Schur form representation of the above matrix, we know that unless $\lambda_1 = 0$, it will not be positive semidefinite. This observation makes the eigenvalues $\{0, \tau_1\alpha_1 - \alpha_0, 1-\tau_1\}$, by inspection, and gives the requirement $\alpha_1 \geq \alpha_0$ as our sufficient condition that $\{x : G_1(x) \geq 0\} \subseteq \{x : G_0(x) \geq 0\}$. A plot of the $G_i(x)$'s for $\alpha_0 = 1$ and $\alpha_1 = 2$ is given in Figure 3.1, which confirms the containment results implied by $\alpha_1 > \alpha_0$.

3.4 Fitting functions with polynomials

A central part of implementing receding horizon control is finding the terminal cost function that gives the worst case cost-to-go using some baseline controller. Typically this function is only known on some grid of the system's states, so that the cost function data will need to be interpolated for any particular state condition that is not exactly on one of the grid points. These worst case cost-to-go functions are all positive semidefinite functions since they represent the cost incurred by worst possible disturbance acting on the system. This semidefiniteness makes it seem reasonable to attempt to fit the terminal cost-to-go data with a positive semidefinite function instead of relying on table interpolation. If we were more concerned with flexibility than positive semidefiniteness, we could also fit the data with general polynomials.

3.4.1 Fitting data to sum-of-squares polynomials

If we restrict the set of positive semidefinite functions that we are considering to be $\Sigma_{n,2p}$, then we can formulate the problem of fitting a function to the data as an LMI. Let the set $\{x_i, y_i\}_{i=1}^m$ with $x_i \in \mathbb{R}^n$, and $y_i \in \mathbb{R}$ be the data that we are attempting to fit. We look to minimize the error of the fit to some $f \in \Sigma_{n,2p}$, and from Lemma 3.3 we know that this function can be represented by at least one matrix $M_p \in \mathbb{R}^{q \times q} \geq 0$, where $q = \sum_{i=0}^{p} \binom{n+i-1}{i}$.

An important question in fitting the data is which error criterion should we use. Is it relative error or absolute error that we really care about? Or, should we use a combination? Here we will consider the following uniform error bound

$$|f(x_i) - y_i| \leq \epsilon \quad i = 1, \ldots, m$$

with $\epsilon \geq 0$. We could switch to a relative error bound by replacing the ϵ in the above equation with ϵy_i.

A sensible optimization problem is presented by the following LMI that drives our data fitting effort for sum-of-squares polynomials

$$\begin{aligned} \min \quad & \epsilon \\ \text{s.t.} \quad & \left| \hat{x}_i^* M_p \hat{x}_i - y_i \right| \leq \epsilon \quad i = 1, \ldots, m \\ & \epsilon \geq 0 \\ & M_p \geq 0 \end{aligned} \tag{3.5}$$

with

$$\hat{x} = \begin{bmatrix} 1 \\ x \\ x^{[2]} \\ \vdots \\ x^{[p]} \end{bmatrix}$$

Since LMI (3.5) is not an LMI in p, a way to solve the fitting problem is to start with a small p and solve (3.5). If the ϵ for this p is too large, then increment p and solve (3.5) again. Repeating these steps until ϵ is small enough forms a reasonable algorithm for fitting data to an $f \in \Sigma_{n,2p}$. If the data comes from a function that is nothing like a sum of squares polynomial, then even for large p the quality of the fit will be low.

Considerations in the number of points to fit

Fitting the data to some $f \in \Sigma_{n,2p}$, is a fit whose parameters are described by a $q \times q$ symmetric matrix, and thus has $\frac{q(q+1)}{2}$ degrees of freedom, where again $q = \sum_{i=0}^{p} \binom{n+i-1}{i}$. But since this matrix is not a unique representation of the polynomial $f \in \Sigma_{n,2p}$, there are really only q degrees of freedom. If $m < q$, the problem will be under constrained and a perfect fit should be achievable. When $m > q$, the problem will be over constrained and a small value for ϵ will be achievable only when the data comes from a function that is close to being in $\Sigma_{n,2p}$, which inversely means that achieving a small value for ϵ for an over constrained problem shows that fitting with a sum-of-squares function was reasonable.

3.4.2 Fitting general polynomials

Removing the positive semidefinite condition on a polynomial fit allows us to search over all functions in $P_{n,p}$ for some degree p. From Lemma 3.1 we know that if p is even, then we can represent it as a 'quadratic' form, which need not be unique. If we were not interested in a 'quadratic' form we can specify any $f \in P_{n,p}$ with a single vector

Lemma 3.4 *Every $f \in P_{n,p}$ can be represented uniquely by a vector $c \in \mathbb{R}^{\sum_{i=0}^{p} \binom{n+i-1}{i}}$.*

Proof:

We can write any $f \in P_{n,p}$ uniquely in terms of each possible monomial as

$$f(x) = \sum_{i=0}^{p} \sum_{j=1}^{\binom{n+i-1}{i}} c_{ij} x_j^{[i]}$$

or

$$f(x) = \sum_{i=0}^{p} c_i^* x^{[i]}$$

with each $c_i \in \mathbb{R}^{\binom{n+i-1}{i}}$. Stacking these vectors on top of each other constructs the c in the Lemma, and using notation as defined for the sum of squares fit we can write the polynomial as $f = c^* \hat{x}$.

□

With this representation of any $f \in P_{n,p}$, we fit the data $\{x_i, y_i\}_{i=1}^m$ with a search for a vector, c, that solves the following equation

$$\begin{bmatrix} -- & \hat{x}_1^* & -- \\ -- & \hat{x}_2^* & -- \\ & \vdots & \\ -- & \hat{x}_m^* & -- \end{bmatrix} c = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \tag{3.6}$$

Since the fit will most likely not be perfect we will try to minimize the error of the fit

$$\begin{bmatrix} -- & \hat{x}_1^* & -- \\ -- & \hat{x}_2^* & -- \\ & \vdots & \\ -- & \hat{x}_m^* & -- \end{bmatrix} c - \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} = e \tag{3.7}$$

If we are interested in minimizing the errors of the fit by minimizing $\|e\|_2$, this is just a least squares problem. If we look to minimize $\|e\|_\infty$, then we can write the fitting problem as a linear program. Also, we could consider a weighted problem where we look to minimize $\|We\|_b$ for some weighting matrix W and some norm b.

Like the sum of squares fit, this general polynomial fit requires us to pay some attention to the the number of coefficients that we are fitting to insure that our fit is relevant. Again we have $q = \sum_{i=0}^{p} \binom{n+i-1}{i}$ degrees of freedom, which implies that we should take $m > q$ to have a sensible fit.

Chapter 4

Simultaneous Stability

The question that motivates this chapter is: "Can we demonstrate stability for all of the following systems?"

$$\dot{x}_i = A_i x_i \qquad i = 1, \ldots, s \tag{4.1}$$

with $A_i \in \mathbb{R}^{n \times n}$, $x_i \in \mathbb{R}^n$, and s finite. If the answer to the above question is yes, then the systems could also have a special kind of joint stability, defined below. First, we will define stability with the following theorem.

Theorem 4.1 (adapted from Theorem 1, §9.3 [HirS74]) *Let $\bar{x} \in \mathbb{R}^n$ be an equilibrium of a C^1 map $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $V : U \to \mathbb{R}$ be a continuous function defined on a neighborhood $U \subset \mathbb{R}^n$ of $\bar{x}$, differentiable on $U - \bar{x}$, such that*

1. *$V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$;*

2. *$\dot{V} \leq 0$ in $U - \bar{x}$.*

Then $\bar{x}$ is stable. Furthermore if also the inequality on $\dot{V}$ is strict, then $\bar{x}$ is asymptotically stable. Any V that satisfies these conditions is referred to as a Lyapunov function.

Corollary 4.1 *If we restrict V to be a polynomial and f to be linear we satisfy the technical conditions and can take $U = \mathbb{R}^n$. The Lyapunov function, V, need only be such that*

1. $V(0) = 0$ *and* $V(x) > 0$ *if* $x \neq 0$*;*

2. $\dot{V} \leq 0$ *for* $x \neq 0$.

to guarantee stability.

Now, we have the necessary ideas to define the type of joint stability in which we are interested.

Definition 4.1 (Simultaneous Stability) *A finite collection of dynamical systems,* $\{f_i\}_{i=1}^{s}$ *with* $\dot{x}_i = f_i(x_i)$*, are called* ***simultaneously stable*** *if there exists a single Lyapunov function that demonstrates stability for each of the* s *systems.*

Clearly, if any of the individual systems in (4.1) are not stable, then the collection of systems will not be simultaneously stable. If we restrict our Lyapunov functions to be polynomials and our systems to be linear, as in (4.1), we can establish a sufficient condition for simultaneous stability of the systems by only checking the conditions in the corollary.

4.1 Quadratic Lyapunov functions

Restricting our search to positive definite quadratic candidate Lyapunov functions,

$$V(x) = x^* P x$$

with some $P > 0$, we get the following sufficient condition for simultaneous stability.

Lemma 4.1 *The systems in (4.1) are simultaneously stable, if there exists a feasible solution to the LMI*

$$\begin{aligned} &\exists P > 0 \\ s.t. \quad &A_i^* P + P A_i \leq 0 \quad i = 1, \dots, s \end{aligned} \tag{4.2}$$

Moreover, the Lyapunov function that demonstrates the simultaneous stability is given by $V(x) = x^* P x$ *with* P *any feasible solution of (4.2).*

Proof:

Form the Lyapunov function $V(x) = x^*Px$, with P any feasible solution of (4.2). By P's being a feasible solution of (4.2), we know that $V(x)$ is positive definite. The derivative of V along the trajectories of each of the s systems in (4.1) is of the form

$$\dot{V} = \dot{x}^*Px + x^*P\dot{x} = x^*\left(A_i^*P + PA_i\right)x$$

which is negative semidefinite by LMI (4.2) for $i = 1, \ldots, s$. Thus, the constructed quadratic Lyapunov function demonstrates stability for all systems in (4.1), implying that the systems are simultaneously stable.

□

If no feasible solution is found for (4.2), then we have shown that there exists no positive definite quadratic Lyapunov function that demonstrates simultaneous stability for all of the systems in (4.1), but we have not shown that the set of systems is not simultaneously stable. So, we must look to other Lyapunov functions to demonstrate simultaneous stability.

4.2 Homogeneous sum of squares Lyapunov functions

Generalizing the idea from the previous section we can attempt to find Lyapunov functions that are polynomials with degree greater than two. Taking $V(x) \in \Sigma_{n,2p} \cap H_{n,2p}$ we have homogeneous sum of squares polynomials of degree $2p$ as our Lyapunov functions. By the definition of the sets of polynomials we know that $V(x) \geq 0$ and $V(0) = 0$ for $p \geq 1$. Lemmas 3.2 and 3.3 tell us that we can write V with some $M \geq 0$ as

$$V(x) = x^{[p]*}Mx^{[p]} \tag{4.3}$$

which is not a valid candidate Lyapunov function since it is only positive semidefinite. We can get the required definiteness by considering only V's that have $M > 0$ or V's that are everywhere no smaller than some positive definite function. Requiring M to be positive definite is more restrictive, so we will take the approach of requiring $V(x) \geq g(x)$ with $g(x) > 0$.

Definition 4.2 *Define the set*

$$\tilde{\mathcal{I}}_p := \left\{ X \geq 0 \in I\!R^{\binom{n+p-1}{p} \times \binom{n+p-1}{p}} : x^{[p]*} X x^{[p]} = \sum_{i=1}^{n} x_i^{2p} \right\}$$

which gives positive definite functions $g(x) = x^{[p]*} \tilde{I}_p x^{[p]} = \sum_{i=1}^{n} x_i^{2p} > 0$ *for* $\tilde{I}_p \in \tilde{\mathcal{I}}_p$.

Remark: The choice of notation for $\tilde{\mathcal{I}}_p$ becomes clearer when we notice that $\tilde{\mathcal{I}}_1 = \{I_{n \times n}\}$, the n by n identity matrix.

The previous definition sets up the following lemma

Lemma 4.2 *Take* $g(x) = x^{[p]*} \tilde{I}_p x^{[p]}$ *and* $V(x) = x^{[p]*} M x^{[p]}$ *with* $M \geq 0$. *If* $M - \epsilon \tilde{I}_p \geq 0$ *for any* $\epsilon > 0$ *and at least one* $\tilde{I}_p \in \tilde{\mathcal{I}}_p$, *then* $V(x) > 0$.

Proof:

$M - \epsilon \tilde{I}_p \geq 0$ implies

$$\begin{aligned} x^{[p]*} \left[M - \epsilon \tilde{I}_p \right] x^{[p]} \geq 0 \quad &\Rightarrow \quad x^{[p]*} M x^{[p]} \geq \epsilon\, x^{[p]*} \tilde{I}_p x^{[p]} \\ &\Rightarrow \quad V(x) \geq \epsilon g(x) \end{aligned}$$

By the definition of $\tilde{\mathcal{I}}_p$ we can see that $g(x) = \sum_{i=1}^{n} x_i^{2p}$, which implies that $g(x) > 0$, $\forall x \neq 0$, making $V(x)$ positive definite.

□

The functions g that are described by $\tilde{\mathcal{I}}_p$ are far from being all the positive definite functions on x, but this condition is much more relaxed than requiring $M > 0$ and it also fits into our LMI frame work.

Now that we have a valid candidate Lyapunov function we can give an LMI sufficient condition for simultaneous stability that parallels and actually includes Lemma 4.1.

Lemma 4.3 *The systems in (4.1) are simultaneously stable, if there exists a feasible solution to the LMI*

$$
\begin{array}{rrcll}
 & \exists M & \geq & 0 & \\
 & \exists \tau_{ij} & & & (i,j) \in (\{1,\ldots,s\} \times \{1,\ldots,q_{n,p}\}) \\
s.t. & M - \epsilon \tilde{I}_p & \geq & 0 & \\
 & \Big(A_i\Big)_{[p]}^{*} M + M\Big(A_i\Big)_{[p]} + \sum_{j=1}^{q_{n,p}} \tau_{ij} Q_j & \leq & 0 & i = 1,\ldots,s
\end{array}
\tag{4.4}
$$

for some $\epsilon > 0$*, at least one* $\tilde{I}_p \in \tilde{\mathcal{I}}_p$*, and the* Q_j*'s as defined in §3.2.1. Moreover, the Lyapunov function that demonstrates the simultaneous stability is given by* $V(x) = x^{[p]*} M x^{[p]}$ *with* M *any feasible solution of (4.4).*

Proof:

Form the Lyapunov function $V(x) = x^{[p]*} M x^{[p]}$ with any M that is a feasible solution to (4.4), which via Lemma 4.2 we know is positive definite. Differentiating along trajectories of any of the s systems in (4.1) we get

$$
\begin{aligned}
\dot{V} &= \Big(\dot{x^{[p]}}\Big)^{*} M x^{[p]} + x^{[p]*} M \Big(\dot{x^{[p]}}\Big) \\
&\overset{(a)}{=} \Big(\Big(A_i\Big)_{[p]} x^{[p]}\Big)^{*} M x^{[p]} + x^{[p]*} M \Big(\Big(A_i\Big)_{[p]} x^{[p]}\Big) \\
&= x^{[p]*} \Big(\Big(A_i\Big)_{[p]}^{*} M + M\Big(A_i\Big)_{[p]}\Big) x^{[p]} \\
&= x^{[p]*} \Big(\Big(A_i\Big)_{[p]}^{*} M + M\Big(A_i\Big)_{[p]}\Big) x^{[p]} + \sum_{j=1}^{q_{n,p}} \tau_{ij} \Big(0\Big) \\
&= x^{[p]*} \Big(\Big(A_i\Big)_{[p]}^{*} M + M\Big(A_i\Big)_{[p]}\Big) x^{[p]} + \sum_{j=1}^{q_{n,p}} \tau_{ij} \Big(x^{[p]*} Q_j x^{[p]}\Big) \\
&= x^{[p]*} \Big(\Big(A_i\Big)_{[p]}^{*} M + M\Big(A_i\Big)_{[p]} + \sum_{j=1}^{q_{n,p}} \tau_{ij} Q_j\Big) x^{[p]}
\end{aligned}
\tag{4.5}
$$

where equality (a) comes from Theorem 2.2. $\dot{V}$, as given in the last line of (4.5), is negative semidefinite due to (4.4) for $i = 1, \ldots, s$. Thus, the constructed Lyapunov function demonstrates stability for all systems in (4.1), implying that the systems are simultaneously stable.

□

The addition of the Q_j's to the LMI does not change the Lyapunov function; it just adds degrees of freedom to the LMI which make it feasible for more extreme sets of A_i's.

Like the result in §4.1, the above lemma lets us know if there exists a Lyapunov function of a specific form and is just a sufficient condition for simultaneous stability. This lemma can also be found in a modified form in [Zel94] as Theorem 2.

4.3 General sum of squares Lyapunov functions

The previous section considered sum of squares polynomials as candidate Lyapunov functions, with the restriction that they were also homogeneous, and it would seem only logical that by further expanding our class of candidate functions to non-homogeneous sum of squares polynomials we could prove simultaneous stability for sets of linear systems that failed the feasibility problem of (4.4). Unfortunately this turns out not to be true in the sense of the following definition and theorem.

Definition 4.3 (Candidate Sum of Squares Lyapunov Functions) *Define the set*

$$\mathcal{C}_p := \left\{ f = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[p]} \end{bmatrix} M \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{[p]} \end{bmatrix}^* \in \Sigma_{n,2p} : f(0) = 0 \;\; \text{and} \; \exists \epsilon > 0 : M \geq \epsilon \begin{bmatrix} 0 & & \\ & \tilde{I}_l & \\ & & 0 \end{bmatrix} \right\} \tag{4.6}$$

with $1 \leq l \leq p$ and $\tilde{I}_l \in \tilde{\mathcal{I}}_l$, as all functions in $\Sigma_{n,2p}$ that meet our definiteness criteria to be considered candidate Lyapunov functions.

Similar to §4.2, the set defined in (4.6) does not contain all positive definite sum of squares polynomials of degree $2p$, but it does contain many more than those polynomials in $\Sigma_{n,2p}$ which can be written with a positive definite matrix via Lemma 3.3

Theorem 4.2 *If there exists no Lyapunov function $V \in \mathcal{C}_p$ that demonstrates simultaneous stability for the set of systems in (4.1), then, if there exists a $V \in \mathcal{C}_{p+1}$ that demonstrates simultaneous stability for (4.1) there also exists a $V \in \mathcal{C}_{p+1} \cap H_{n,2(p+1)}$ that does the same.*

Remark: If we were looking for a minimum degree sum of squares Lyapunov function that meets the definiteness constraints of $\mathcal{C}_p$ and demonstrates simultaneous stability for (4.1), then we would only need to check the homogeneous ones following Lemma 4.3 for increasing p.

Before we can prove Theorem 4.2, we need the following lemma and definitions

Lemma 4.4 *If the matrix $M \geq 0$ is partitioned as*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}$$

and $M_{11} = 0$, then $M_{12} = 0$ and $M_{22} \geq 0$.

Proof:

First, we know that since $M \geq 0$, all symmetric blocks along the diagonal are also positive semidefinite, implying $M_{22} \geq 0$. For $M_{12} = 0$, construct the Schur form of M

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^* & U_{22} \end{bmatrix}^* \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^* & U_{22} \end{bmatrix}$$

Since $M_{11} = 0$, there are enough zero eigenvalues to make $D_{11} = 0$ which forces $M_{12} = 0$.

□

Definition 4.4 *For ease of notation, define*

$$\hat{x}^{[p]} := \begin{bmatrix} x \\ x^{[2]} \\ \vdots \\ x^{[p]} \end{bmatrix} \qquad \hat{M}_{[p]} := \begin{bmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,p} \\ M_{1,2}^* & M_{2,2} & & M_{2,p} \\ \vdots & & \ddots & \vdots \\ M_{1,p}^* & M_{2,p}^* & \dots & M_{p,p} \end{bmatrix}$$

and

$$\hat{A}_{[p]} := \begin{bmatrix} A & & & \\ & A_{[2]} & & \\ & & \ddots & \\ & & & A_{[p]} \end{bmatrix}$$

with $\hat{Q}^{[k]}_{[p],j}$ *defined so that*

$$\hat{x}^{[p]*}\hat{Q}^{[k]}_{[p],j}\hat{x}^{[p]} := x^{[k]*}Q_j x^{[k]} = 0$$

Now with Lemma 4.4 and the extended definitions above, we can attack the proof of Theorem 4.2 in an orderly fashion.

Proof: Theorem 4.2

Looking first at $\mathcal{C}_p$ we remember that due to Lemma 3.3 we can write any $f \in \Sigma_{n,2p}$ as

$$f(x) = \begin{bmatrix} 1 \\ x \\ x^{[2]} \\ \vdots \\ x^{[p]} \end{bmatrix}^* \begin{bmatrix} M_{0,0} & M_{0,1} & M_{0,2} & \dots & M_{0,p} \\ M_{0,1}^* & M_{1,1} & M_{1,2} & & M_{1,p} \\ M_{0,2}^* & M_{1,2}^* & M_{2,2} & & M_{2,p} \\ \vdots & & & \ddots & \vdots \\ M_{0,p}^* & M_{1,p}^* & M_{2,p}^* & \dots & M_{p,p} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^{[2]} \\ \vdots \\ x^{[p]} \end{bmatrix}$$

with the restriction from $\mathcal{C}_p$ that $f(0) = 0$, we know that $M_{0,0} = 0$, which with Lemma 4.4 gives

$$f(x) = \hat{x}^{[p]*}\hat{M}_{[p]}\hat{x}^{[p]}$$

and

$$\hat{M}_{[p]} \geq \epsilon \begin{bmatrix} 0 & & \\ & \tilde{I}_l & \\ & & 0 \end{bmatrix}$$

for some $\epsilon > 0$, $1 \leq l \leq p$, and $\tilde{I}_l \in \tilde{\mathcal{I}}_l$ imply that f is a valid candidate Lyapunov function.

With these results we can restate the non-existence of a Lyapunov function in $\mathcal{C}_p$ that demonstrates simultaneous stability for (4.1) as the infeasibility of the following LMI

$$\begin{array}{rcl} \exists \hat{M}_{[p]} & \geq & 0 \\ \exists \tau_{ijk} & & \\ \text{s.t.} \qquad \hat{M}_{[p]} - \epsilon \begin{bmatrix} 0 & & \\ & \tilde{I}_l & \\ & & 0 \end{bmatrix} & \geq & 0 \\ \hat{A}^*_{[p]i}\hat{M}_{[p]} + \hat{M}_{[p]}\hat{A}_{[p]i} + \sum_{k=1}^{p}\sum_{j=1}^{q_{n,k}} \tau_{ijk}\hat{Q}^{[k]}_{[p],j} & \leq & 0 \quad i = 1, \ldots, s \end{array} \tag{4.7}$$

with $(i, j, k) \in \{1, \ldots, s\} \times \{1, \ldots, q_{n,k}\} \times \{1, \ldots, p\}$, $\epsilon > 0$, some $l \in \{1, \ldots, p\}$, and $\tilde{I}_l \in \tilde{\mathcal{I}}_l$.

Looking for a Lyapunov function in $\mathcal{C}_{p+1}$ is the search for a feasible point for the following LMI

$$\begin{array}{rcl} \exists \hat{M}_{[p+1]} & \geq & 0 \\ \exists \tau_{ijk} & & \\ \text{s.t.} \qquad \hat{M}_{[p+1]} - \epsilon \begin{bmatrix} 0 & & \\ & \tilde{I}_l & \\ & & 0 \end{bmatrix} & \geq & 0 \\ \hat{A}^*_{[p+1]i}\hat{M}_{[p+1]} + \hat{M}_{[p+1]}\hat{A}_{[p+1]i} + \sum_{k=1}^{p+1}\sum_{j=1}^{q_{n,k}} \tau_{ijk}\hat{Q}^{[k]}_{[p+1],j} & \leq & 0 \quad i = 1, \ldots, s \end{array} \tag{4.8}$$

with $(i, j, k) \in \{1, \ldots, s\} \times \{1, \ldots, q_{n,k}\} \times \{1, \ldots, p+1\}$, $\epsilon > 0$, some $l \in \{1, \ldots, p+1\}$, and $\tilde{I}_l \in \tilde{\mathcal{I}}_l$.

We know that if LMI (4.8) has a feasible solution it must be of the form

$$\hat{M}_{[p+1]} = \left[\begin{array}{ccc|c} & & & M_{1,p+1} \\ & \hat{M}_{[p]} & & M_{2,p+1} \\ & & & \vdots \\ \hline M^*_{1,p+1} & M^*_{2,p+1} & \ldots & M_{p+1,p+1} \end{array}\right] \geq 0,$$

with

$$\hat{M}_{[p+1]} - \epsilon \begin{bmatrix} 0 & & \\ & \tilde{I}_l & \\ & & 0 \end{bmatrix} \geq 0$$

Notice that (4.7) is almost a sub-problem within (4.8), and if (4.8) is feasible while (4.7) is not, then for a feasible solution of (4.8) either $\hat{M}_{[p]} =$

0, or

$$\hat{M}_{[p+1]} - \epsilon \begin{bmatrix} 0 & & \\ & 0 & \\ & & \tilde{I}_{p+1} \end{bmatrix} \geq 0 \qquad (4.9)$$

is the only valid definiteness constraint.

If $\hat{M}_{[p]} = 0$, then via Lemma 4.4, $V \in \mathcal{C}_{p+1}$ is given by

$$V(x) = x^{[p+1]*} M_{p+1,p+1} x^{[p+1]}$$

which takes us to Lemma 4.3 as the Theorem claimed.

If $\hat{M}_{[p]} \neq 0$, then in order for the last set of constraints in (4.8) to be feasible its diagonal entry of

$$A_{[p+1]i}^{*} M_{p+1,p+1} + M_{p+1,p+1} A_{[p+1]i} + \sum_{j=1}^{q_{n,p+1}} \tau_{i,j,p+1} Q_j \leq 0$$

must also be feasible, and the definiteness requirement of (4.9) forces

$$M_{p+1,p+1} - \epsilon \tilde{I}_{p+1} \geq 0$$

which are again the constraints for Lemma 4.3.

□

4.4 Examples

Looking to examples, we will now show how a useful model fits in our framework of (4.1).

Lemma 4.5 *If the set of systems* $\{A_i\}_{i=1}^{s}$ *is simultaneously stable, then the time varying linear system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \\ A(t) \in \mathcal{A} &:= \mathbf{Co}\{A_1, \ldots, A_s\} = \left\{ \sum_{i=1}^{s} a_i(t) A_i : 0 \leq a_i(t) \leq 1 \ \sum_{i} a_i(t) = 1, \ \forall t \right\} \end{aligned} \qquad (4.10)$$

is stable as long the $a_i(t)$*'s have only a finite number of discontinuities on finite intervals.*

Proof:

Use the Lyapunov function that demonstrated the A_i's simultaneous stability to show the stability of linear time varying system in (4.10).

□

4.4.1 Robustness Bounds

The following example for robustness bounds of a linear time varying system originally appeared in [Zel94]. We would like to know the maximum value of k for which the following system is stable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2-u(t) & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.11}$$

with $0 \leq u(t) \leq k$. We can rewrite this problem as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left((1-a(t)) \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} + a(t) \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.12}$$

with $0 \leq a(t) \leq 1$, or in the form of Lemma 4.5

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A(t) \in \mathcal{A} := \mathbf{Co} \left\{ \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix} \right\} \tag{4.13}$$

which gives a sufficient condition for stability as the simultaneous stability of the two matrices above. The LMI feasibility problem associated with Lemma 4.3 for (4.13) is quasi-convex in k, so, for a fixed p, we can do a bisection on k to find the largest value that makes the LMI feasible. In [Zel94], $p = 1$ (quadratic), and $p = 2$ (quartic) sum of square Lyapunov functions are considered and largest k for which he can demonstrate stability is 5.73. [XieSF97] considers the same example, but approaches the problem with Lyapunov functions that are piecewise maximums of quartic sum of squares polynomials, and finds a maximum value of k of 6.2.

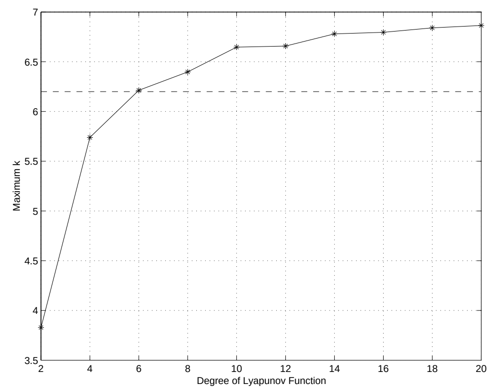

Figure 4.1: Maximal values of k for which we can demonstrate stability of the time varying system in (4.13) as a function of the degree of the sum of squares Lyapunov function. The dashed line gives the maximum value of k reported in [XieSF97].

The results of taking larger values for p are shown in Figure 4.1, which gives the maximum value of k for which we can demonstrate stability of the system in (4.13) as a function of the degree of the sum of squares Lyapunov function. Picking an arbitrary largest value for p of 10, we can demonstrate stability for (4.13) with $k \leq 6.86$.

The ideas in this example easily extend to a larger number of perturbations to the plant, but with more than one perturbation, the bisection technique will no longer work.

4.4.2 Intermittent information observer

Consider now the problem of building an observer for system that has two outputs y_0, and y_1, where we are always allowed to use y_0 for the observer, but we only receive y_1 intermittently. An observer for platoon activity can be cast into this form by letting all the on-board sensors be y_0 and letting all the sensor data that comes from all the other vehicles be y_1. y_1 is subject to some non-trivial packet loss since it needs to be transmitted over a possibly very noisy radio link, and can thus be considered intermittent. The question of showing that this observer is stable can be cast into the form of Lemma 4.5 by looking at the error dynamics of the observer. First, we will write the plant as

$$\begin{bmatrix} \dot{x} \\ y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C_0 & D_0 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

and the observer as

$$\dot{\hat{x}} = A\hat{x} + Bu - K_0\Big(y_0 - C_0\hat{x} - D_0u\Big) - a(t)K_1\Big(y_1 - C_1\hat{x} - D_1u\Big)$$

where $a(t) \in \{0, 1\}$ tells us if y_1 is being received or not. If we define the observer error as $e = x - \hat{x}$, we find it evolves as

$$\dot{e} = Ae - K_0C_0e - a(t)K_1C_1e \tag{4.14}$$

or in the form of Lemma 4.5

$$\dot{e} = \Big[a(t)\Big(A - K_0C_0 - K_1C_1\Big) + (1 - a(t))\Big(A - K_0C_0\Big)\Big]\, e \tag{4.15}$$

So, in order to establish stability of this observer's error dynamics we need only follow Lemma 4.3 for the matrices above, as long as $a(t)$ has only finite discontinuities on finite intervals, which hopefully any transmission network would. It is important to note that in the framework of Lemma 4.3, we will be solving this problem for all possible signals $a(t)$ which will add an additional layer of conservatism to our stability results.

Chapter 5

Conclusions

The power transformation that forms the heart of this report has been around for almost thirty years, but due to its complexity and the non-uniqueness it instills in the matrix forms of polynomials, it has seen only sporadic use. The transform proves to be very useful in that it allows us to form a generalized $\mathcal{S}$-procedure easily and also it makes considering higher than quadratic order polynomials as Lyapunov functions easier.

The most promising directions for extending the results of this report look to be in the application of these methods to actual problems. The sum of squares Lyapunov approach is valid for many other problems than the two considered examples and could become a very useful way to investigate stability of linear time varying systems, since many time varying systems can be formulated as time varying convex combinations of a finite set of linear systems.

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