An LMI method to demonstrate simultaneous stability using non-quadratic polynomial Lyapunov functions

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Abstract

We consider a nonlinear state transformation that allows us to work with non-quadratic polynomial Lyapunov functions. We use these polynomials to form Lyapunov functions to demonstrate simultaneous stability for a finite collection of linear systems. Under a weak definiteness condition, our main result, Theorem 3, shows that the minimum degree polynomial Lyapunov function that demonstrates simultaneous stability for a collection of linear systems can be written as a homogeneous polynomial.

1 Introduction

This paper is based around using a nonlinear state transform to study and exploit the characteristics of multivariable higher degree polynomials with linear algebra.

By working with the polynomials in the familiar forms of vectors and matrices, we can expand results relating to quadratic polynomials to ones of arbitrary degree. The familiarity that we gain with high degree polynomials under the nonlinear state transformation, does come at the price of unique representations.

Even with the troubles of non-uniqueness, this transformation allows us to to form the central result of Section 4, that limits the size of the search for a Lyapunov function for simultaneous stability, as an LMI feasibility problem. The convexity of such a search is a very pleasing byproduct of using the transform, and the problem formulation allows us to easily consider a large set of examples. The use of this transformation to study Lyapunov functions to demonstrate simultaneous stability was pioneered in [10].

2 A power transformation

Definition 1 *A **monomial** is a product of variables raised to non-negative integer powers. For example*

$$x^2 \qquad and \qquad x^3yz^{10}$$

are both monomials.

Definition 2 *The **degree** of a monomial is the sum of the powers to which each variable is raised. For example, x^2 has degree 2 while x^3yz^{10} has degree 14.*

Definition 3 *Consider a vector $x \in \mathbb{R}^n$. The **power transformation of degree** p introduced in [4] is a nonlinear change of coordinates that forms a new vector $x^{[p]}$ of all monomials of degree p that can be made from the original x vector,*

$$x_l^{[p]} := x_1^{p_{l1}} x_2^{p_{l2}} \cdots x_n^{p_{ln}} \qquad l = 1, \cdots, m$$

with $m = \binom{n+p-1}{p}$ ordered by the degree of the leftmost variable.

Note that with this definition $x^{[0]} = 1$, and $x^{[1]} = x$. The following are examples of illustrate the results of the transform for more complicated cases

1. The most basic form is $n = p = 2 \quad \Rightarrow \quad m = 3$
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow \quad x^{[2]} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}$$
2. $n = 3, p = 3 \quad \Rightarrow \quad m = 10$
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \Rightarrow \quad x^{[3]} = \begin{bmatrix} x_1^3 \\ x_1^2x_2 \\ x_1^2x_3 \\ x_1x_2^2 \\ x_1x_2x_3 \\ x_1x_3^2 \\ x_2^3 \\ x_2^2x_3 \\ x_2x_3^2 \\ x_3^3 \end{bmatrix}$$

2.1 Linear ODEs under the transform

The power transformation preserves linear ODE relations which makes the transform useful for studying linear systems.

Theorem 1 *[4] Given $A(t) \in \mathbb{R}^{n\times n}$ and $p \in \mathbb{Z}_+$, with $\dot{x}(t) = A(t)x(t)$, then there exists a matrix $A_{[p]}(t) \in \mathbb{R}^{\binom{n+p-1}{p}\times\binom{n+p-1}{p}}$ such that $\frac{d}{dt}\left(x^{[p]}(t)\right) = A_{[p]}(t)x^{[p]}(t)$, and the operation that maps A to $A_{[p]}$ is linear.*

To clarify the meaning of $A_{[p]}$, consider the following example where $n = p = 2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow A_{[2]} = \begin{bmatrix} 2a_{11} & 2a_{12} & 0 \\ a_{21} & a_{11}+a_{22} & a_{12} \\ 0 & 2a_{21} & 2a_{22} \end{bmatrix}$$

We can augment this theorem with the following result also from [4] that shows that stability of a linear ODE is preserved under the transform.

Lemma 1 *If A is Hurwitz, then $A_{[p]}$ is as well.*

See [1], [10] for further developments along these lines.

3 Quadratic form representation of polynomials using the power transform

The previous section provides a useful tool which allows us to work with high degree polynomials in the familiar setting of quadratic functions on matrices. Before studying representations of polynomials under the power transformation, we make the following definitions.

Definition 4 *A* ***polynomial*** *is a sum of monomials, and the degree of a polynomial is given by the maximum degree of its monomials. Let $P_{n,d}$ be the set of all polynomials in n variables with degree less than or equal to d.*

Definition 5 *Let $H_{n,d}$ be the set of all* ***homogeneous polynomials*** *in n variables with degree d. $f \in H_{n,d}$ iff $f \in P_{n,d}$ and $f(\lambda x) = \lambda^d f(x)$.*

For notation define

$$\hat{x}^{[p]} := [x, x^{[2]}, \ldots, x^{[p]}]^*$$

Lemma 2 *$f \in P_{n,2t}$ if and only if $\exists M \in \mathbb{R}^{q \times q}$, with $q = \sum_{i=0}^{t} \binom{n+i-1}{i}$ such that $M = M^*$ and*

$$f(x) = \begin{bmatrix} 1 \\ \hat{x}^{[t]} \end{bmatrix}^* \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} 1 \\ \hat{x}^{[t]} \end{bmatrix}.$$

Proof: $\Leftarrow$ Working out the multiplication gives a polynomial in n variables of degree $2t$.

$\Rightarrow$ All the monomials of $f \in P_{n,2t}$ can be factored into products of two monomials of degree less than or equal to t, so there is at least one, not necessarily symmetric, matrix R such that

$$f(x) = \begin{bmatrix} 1 \\ \hat{x}^{[t]} \end{bmatrix}^* \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} 1 \\ \hat{x}^{[t]} \end{bmatrix}$$

Then take $M = \frac{1}{2}(R + R^*)$. ■

Lemma 3 *$f \in H_{n,2p}$ iff $\exists M = M^* \neq 0$ such that $f(x) = x^{[p]*} M x^{[p]}$.*

Proof: Follow the same steps as for Lemma 2 ■

Remark: Lemma 2 is equivalent to the statement "All polynomials in x of degree $2t$ can be represented as a symmetric quadratic form in $[1, \hat{x}^{[t]}]^*$."

This lemma works on odd degree polynomials as well by picking $2t$ to be greater than the degree. The following example shows one representation for an $f \in P_{1,4}$

$$\begin{aligned} f(x) &= x^4 + 2x^3 + 2x^2 + 1 \\ &= \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \end{aligned}$$

From Lemma 2 and the example we can see that if the matrix inside the quadratic form is positive (semi)definite, then the polynomial is positive (semi)definite. All polynomials that can be factored into sums of squares of other polynomials, referred to as sum-of-squares polynomials, can be put in this quadratic form with a positive semidefinite matrix as is shown in [7], where the matrix is called the "Gram" matrix. Unfortunately, there exist positive semidefinite polynomials that can not be factored into this form with a semidefinite matrix.

Additionally there is the problem of the non-uniqueness of the matrix representations which was recognized in [2] and was heavily investigated along with sum-of-squares polynomials in [6].

We can write a given homogeneous polynomial as $f = x^{[p]*} Q x^{[p]}$. To account for the non-uniqueness, we take Q_0 as a particular representation of the polynomial, such that $f = x^{[p]*} Q_0 x^{[p]}$, and then we find all the symmetric matrices, Q_i^p, defined such that $x^{[p]*} Q_i^p x^{[p]} = 0$, $\forall x$. These Q_i^p's can be thought of as spanning the null space of this operator. With these matrices we can parameterize the polynomial as

$$f(x) = x^{[p]*} \left[Q_0 + \sum_i \lambda_i Q_i^p \right] x^{[p]}.$$

If we are interested in knowing if this polynomial is sum-of-squares we look to find if there exists at least one $Q \geq 0$, and, nicely, this is just a LMI feasibility problem in the λ_i's [7].

3.1 The Q_i^p's

Why should there be any other matrix aside from the zero matrix such that $x^{[p]*} Q_i^p x^{[p]} = 0$? If we look back to the definition of the power transform, we see that the elements are not independent. Considering $n = p = 2$, from the first example in §2, we have $(x_1^2)(x_2^2) = (x_1 x_2)^2$. This relation gives the Q_1^2 matrix as

$$Q_1^2 = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$$

For $n = p = 2$ this is the only Q_i^p matrix. For any p, n we will denote the number of these matrices $q_{n,p}$.

4 Simultaneous Stability

The question that motivates the previous developments is: "Can we demonstrate stability for all of the following systems?"

$$\dot{x}_i = A_i x_i \qquad i = 1, \ldots, s \tag{1}$$

with $A_i \in \mathbb{R}^{n\times n}$, $x_i \in \mathbb{R}^n$, and s finite. If the answer to the above question is yes, then the systems could also have a special kind of joint stability, defined below. First, we will define stability and a way of demonstrating it.

Definition 6 *An equilibrium point, $\bar{x}$, of a differential equation is **stable** if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|x(t_0) - \bar{x}\| \le \delta$ then $\|x(t) - \bar{x}\| \le \epsilon$, $\forall t > 0$.*

Theorem 2 (adapted from Theorem 1, §9.3 [5]) *Let $\bar{x} \in \mathbb{R}^n$ be the equilibrium of a C^1 differential equation on $\mathbb{R}^n$. Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a continuous function differentiable on $\mathbb{R}^n - \bar{x}$, s.t.*

1. *$V(\bar{x}) = 0$ and $V(x) > 0$ if $x \ne \bar{x}$;*
2. *$\dot{V} \le 0$ on $\mathbb{R}^n - \bar{x}$.*

Then $\bar{x}$ is stable. Furthermore if also the inequality on $\dot{V}$ is strict, then $\bar{x}$ is asymptotically stable. Any V that satisfies these conditions is referred to as a Lyapunov function.

Now, we have the necessary ideas to define the type of joint stability in which we are interested.

Definition 7 *A finite collection of dynamical systems, $\{A_i\}_{i=1}^s$ with $\dot{x}_i = A_i x_i$, are called **simultaneously stable** if there exists a single Lyapunov function that demonstrates stability for each of the s systems.*

Clearly, if any of the individual systems, the A_i's, is not stable, then the collection of systems will not be simultaneously stable. If we restrict our Lyapunov functions to be polynomials, we can establish sufficient conditions for simultaneous stability of the systems as LMI problems.

4.1 Quadratic Lyapunov functions

Restricting our search to positive definite quadratic candidate Lyapunov functions,

$$V(x) = x^* P x$$

with some $P > 0$, we get the following standard sufficient condition for simultaneous stability, which is also referred to as "Quadratic Stability" in [3].

Lemma 4 *The systems in (1) are simultaneously stable, if there exists a feasible solution to the LMI*

$$\begin{array}{c} \exists P > 0 \\ s.t. \quad A_i^* P + P A_i \le 0 \quad i = 1, \ldots, s \end{array} \tag{2}$$

Moreover, the Lyapunov function that demonstrates the simultaneous stability is given by $V(x) = x^* P x$ with P any feasible solution of (2).

Proof: Form the Lyapunov function $V(x) = x^* P x$, with P any feasible solution of (2). Since P is a feasible solution of (2), we know that $V(x)$ is positive definite. The derivative of V along the trajectories of each of the s systems in (1) is of the form

$$\dot{V} = \dot{x}^* P x + x^* P \dot{x} = x^* \Big(A_i^* P + P A_i \Big) x$$

which is negative semidefinite by LMI (2) for $i = 1, \ldots, s$. Thus, the constructed quadratic Lyapunov function demonstrates stability for all systems in (1), implying that they are simultaneously stable. ■

If no feasible solution is found for (2), then we have shown that there exists no positive definite quadratic Lyapunov function that demonstrates simultaneous stability for all of the systems in (1), but we have not shown that the set of systems is not simultaneously stable. So, we must look to other Lyapunov functions to demonstrate simultaneous stability.

4.2 Homogeneous Lyapunov functions

The idea from the previous section was generalized in [10] in an attempt to find Lyapunov functions that are polynomials with degree greater than two. Taking $V(x) \in H_{n,2p}$, and $M_p = M_p^*$, we have

$$V(x) = x^{[p]*} M_p x^{[p]} \tag{3}$$

which is not a valid candidate Lyapunov function since it is not necessarily positive definite. We can get the required definiteness by considering only V's that have $M_p > 0$, as is implicitly done in [10], or V's that are everywhere no smaller than some positive definite function. We will take the less restrictive approach by requiring $V(x) \ge g(x)$ with $g(x) > 0$.

Lemma 5 *If the following LMI is feasible for any $\epsilon > 0$, then $V(x) = x^{[p]*} M_p x^{[p]}$ is positive definite.*

$$\begin{array}{rcl} \exists \tilde{I}_p & = & \tilde{I}_p^* \\ \exists M_p & = & M_p^* \\ s.t. & & \\ x^{[p]*} \tilde{I}_p x^{[p]} & = & \sum_{k=1}^n x_k^{2p} \\ M_p - \epsilon \tilde{I}_p & \ge & 0 \end{array}$$

Proof: $M_p - \epsilon\, \tilde{I}_p \ge 0$ implies

$$x^{[p]*} \Big[M_p - \epsilon\, \tilde{I}_p \Big] x^{[p]} \ge 0 \;\Rightarrow\; x^{[p]*} M_p x^{[p]} \ge \epsilon\, x^{[p]*} \tilde{I}_p x^{[p]}$$

$$\Rightarrow\; V(x) \ge \epsilon \sum_{k=1}^n x_k^{2p}$$

$\sum_{k=1}^n x_k^{2p}$ is positive definite, thus $V(x)$ is as well. ■

There exist positive definite polynomials that do not satisfy the LMI conditions above, but these conditions are less restrictive than $M_p > 0$ and they also fit into the LMI frame work.

With this valid candidate Lyapunov function we can pose an LMI sufficient condition for simultaneous stability that parallels and actually includes Lemma 4.

Lemma 6 *The systems in (1) are simultaneously stable, if there exists a feasible solution to the LMI*

$$\begin{aligned} \exists M_p &= M_p^* \\ \exists \tilde{I}_p &= \tilde{I}_p^* \\ \exists \tau_{ij} & \\ s.t. & \\ \sum_{k=1}^{n} x_k^{2p} &= x^{[p]*}\tilde{I}_p x^{[p]} \\ 0 &\leq M_p - \epsilon\, \tilde{I}_p \\ 0 &\geq \Big(A_i\Big)_{[p]}^* M_p + M_p\Big(A_i\Big)_{[p]} + \sum_{j=1}^{q_{n,p}} \tau_{ij} Q_j^p \end{aligned} \tag{4}$$

for $(i, j) \in (\{1, \ldots, s\} \times \{1, \ldots, q_{n,p}\})$, some $\epsilon > 0$, and the Q_j^p's as defined in §3.1. Moreover, the Lyapunov function that demonstrates the simultaneous stability is given by $V(x) = x^{[p]*}M_p x^{[p]}$.

Proof: Form the Lyapunov function $V(x) = x^{[p]*}M_p x^{[p]}$ with M_p a feasible solution to (4), which via Lemma 5 we know is positive definite. Differentiating along trajectories of any of the s systems in (1) we have

$$\begin{aligned} \dot{V} &= \left(\dot{x^{[p]}}\right)^* M_p x^{[p]} + x^{[p]*} M_p \left(\dot{x^{[p]}}\right) \\ &\overset{(a)}{=} \left(\Big(A_i\Big)_{[p]} x^{[p]}\right)^* M_p x^{[p]} + x^{[p]*} M_p \left(\Big(A_i\Big)_{[p]} x^{[p]}\right) \\ &= x^{[p]*}\left(\Big(A_i\Big)_{[p]}^* M_p + M_p\Big(A_i\Big)_{[p]}\right) x^{[p]} + \sum_{j=1}^{q_{n,p}} \tau_{ij}\Big(0\Big) \\ &= x^{[p]*}\left(\Big(A_i\Big)_{[p]}^* M_p + M_p\Big(A_i\Big)_{[p]} + \sum_{j=1}^{q_{n,p}} \tau_{ij} Q_j^p\right) x^{[p]} \end{aligned} \tag{5}$$

where equality (a) comes from Theorem 1. $\dot{V}$, in the last line of (5), is negative semidefinite due to (4) for $i = 1, \ldots, s$. Thus, the constructed Lyapunov function demonstrates stability for all systems in (1), implying that the systems are simultaneously stable. ■

The addition of the Q_j^p's to the LMI does not change the Lyapunov function; it adds degrees of freedom to the LMI to make it feasible for more extreme A_i's. As with §4.1, this result lets us know if there exists a Lyapunov function of a specific form and is only a sufficient condition for simultaneous stability. This result is found, with stricter positive definite requirements on V, as Theorem 2 in [10].

4.3 General polynomial Lyapunov functions

The previous section considered sum of squares polynomials as candidate Lyapunov functions, with the restriction that they were also homogeneous, and it would seem only logical that by further expanding our class of candidate functions to non-homogeneous polynomials we could prove simultaneous stability for sets of linear systems that failed the feasibility problem of (4). Unfortunately this turns out not to be true in the sense of Theorem 3, which we can compactly state after a few definitions and a supporting lemma.

Definition 8 *For ease of notation, define*

$$\hat{M}_p := \begin{bmatrix} M_1 & M_{1,2} & \ldots & M_{1,p} \\ M_{1,2}^* & M_2 & & M_{2,p} \\ \vdots & & \ddots & \vdots \\ M_{1,p}^* & M_{2,p}^* & \ldots & M_p \end{bmatrix}$$

and

$$(\hat{A}_i)_{[p]} := diag(A, A_{[2]}, \ldots, A_{[p]})$$

with $\hat{Q}_j^{[p]}$ defined so that

$$\hat{x}^{[p]*}\hat{Q}_j^p \hat{x}^{[p]} := 0$$

in general there will be $\hat{q}_{n,p}$ of them.

Noting that positive definite polynomials can not have constant or linear terms explains why all the definitions are based on $\hat{x}^{[p]}$. These definitions now allow us to pose the question of whether there exists a non-homogeneous polynomial Lyapunov function that meets our definiteness criteria and demonstrates simultaneous stability for the systems in (1).

Lemma 7 *The systems in (1) are simultaneously stable, if there exists a feasible solution to the LMI*

$$\begin{aligned} \exists \hat{M}_p &= \hat{M}_p^* \\ \exists \tilde{I}_p &= \tilde{I}_p^* \\ \exists \tau_{ij} & \\ s.t. & \\ \sum_{k=1}^{n} x_k^{2p} &= x^{[p]*}\tilde{I}_p x^{[p]} \\ 0 &\leq \hat{M}_p - \epsilon \begin{bmatrix} 0 & \\ & \tilde{I}_p \end{bmatrix} \\ 0 &\geq (\hat{A}_i)_{[p]}^* \hat{M}_p + \hat{M}_p(\hat{A}_i)_{[p]} + \sum_{j=1}^{\hat{q}_{n,p}} \tau_{ij}\hat{Q}_j^p \end{aligned} \tag{6}$$

for $(i, j) \in (\{1, \ldots, s\} \times \{1, \ldots, \hat{q}_{n,p}\})$, some $\epsilon > 0$, and the $\hat{Q}_j^p$'s as defined above. Moreover, the Lyapunov function that demonstrates the simultaneous stability is given by $V(x) = \hat{x}^{[p]*}\hat{M}_p\hat{x}^{[p]}$.

Proof: The proof follows the same steps as the proof for Lemma 6 ■

Theorem 3 *If the LMI (6) is feasible with some symmetric matrix $\hat{M}_p$ such that $\hat{x}^{[p]*}\hat{M}_p\hat{x}^{[p]} \in P_{n,2p}$, then there also exists a symmetric matrix M_p such that $x^{[p]*}M_p x^{[p]} \in H_{n,2p}$ also demonstrates simultaneous stability of the set of systems in (1).*

Remark: If we were looking for a minimum degree polynomial Lyapunov function that demonstrates simultaneous stability for (1), then we would only need to check the homogeneous ones following Lemma 6 for increasing p.

Proof: [Theorem 3] Assume $\hat{M}_p$, $\tilde{I}_p$ and ϵ satisfy LMI (6), then partition

$$\hat{M}_p = \left[\begin{array}{c|c} \hat{M}_{p-1} & \bar{M} \\ \hline \bar{M}^* & M_p \end{array}\right]$$

the first inequality constraint in (6) gives

$$0 \leq \left[\begin{array}{c|c} \hat{M}_{p-1} & \bar{M} \\ \hline \bar{M}^* & \left(M_p - \epsilon\tilde{I}_p\right) \end{array}\right]$$

which implies that

$$0 \leq M_p - \epsilon\tilde{I}_p$$

showing that M_p meets the first inequality of (4).

Partitioning each $\hat{Q}_j^p$ as

$$\left[\begin{array}{c|c} Q_j^{11} & Q_j^{12} \\ \hline Q_j^{12*} & Q_j^{22} \end{array}\right]$$

we can rewrite the second inequality in (6) as

$$0 \geq \left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{B}^* & \mathsf{C} \end{array}\right]$$

where

$$\mathsf{A} = (\hat{A}_i)^*_{[p-1]}\hat{M}_{p-1} + \hat{M}_{p-1}(\hat{A}_i)_{[p-1]} + \sum_{j=1}^{\hat{q}_{n,p}} \tau_{ij} Q_j^{11}$$

$$\mathsf{B} = (\hat{A}_i)^*_{[p-1]}\bar{M} + \bar{M}(A_i)_{[p]} + \sum_{j=1}^{\hat{q}_{n,p}} \tau_{ij} Q_j^{12}$$

$$\mathsf{C} = (A_i)^*_{[p]}M_p + M_p(A_i)_{[p]} + \sum_{j=1}^{\hat{q}_{n,p}} \tau_{ij} Q_j^{22}$$

Since the entire matrix is negative semidefinite, C must be as well. $\mathsf{C} \leq 0$ is almost the second inequality from (4), except the relationship between Q_j^{22} and Q_j^p needs to be clarified. From the partitioning of $\hat{Q}_j^p$, we know that all the terms of degree $2p$ in $\hat{x}^{[p]*}\hat{Q}_j^p\hat{x}^{[p]}$ come from Q_j^{22}, so $x^{[p]*}Q_j^{22}x^{[p]}$ must equal zero. Thus, $\text{span}(Q_j^{22}) \subseteq \text{span}(Q_j^p)$. Now, take any Q_j^p and stack it with zeros to form

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & Q_j^p \end{array}\right]$$

which is a valid $\hat{Q}_j^p$ matrix and shows that $\text{span}(Q_j^p) \subseteq \text{span}(Q_j^{22})$. These containment results illustrate that $\mathsf{C} \leq 0$ is equivalent to the second inequality in (4). Proving that if (6) is feasible, then (4) is as well. ■

5 Examples

Looking to examples, we will now show how a useful model fits in our framework of (1).

Lemma 8 *If the set of systems $\{A_i\}_{i=1}^s$ is simultaneously stable, then the time varying linear system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \\ A(t) &\in \mathcal{A} := \mathbf{Co}\{A_1, \ldots, A_s\} \end{aligned} \tag{7}$$

*where **Co** denotes convex hull, is stable as long the time varying combinations have only a finite number of discontinuities on finite intervals.*

Proof: Use the Lyapunov function that demonstrated the A_i's simultaneous stability to show the stability of linear time varying system in (7). ■

5.1 Robustness Bounds

The following example for robustness bounds of a linear time varying system originally appeared in [10]. We would like to know the maximum value of k for which the following system is stable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2-u(t) & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8}$$

with $0 \leq u(t) \leq k$. We can rewrite this problem as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left((1-a(t))\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} + a(t)\begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $0 \leq a(t) \leq 1$, or in the form of Lemma 8

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= A(t)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ A(t) \in \mathcal{A} &:= \mathbf{Co}\left\{\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix}\right\} \end{aligned} \tag{9}$$

which gives a sufficient condition for stability as the simultaneous stability of the two matrices above. The LMI feasibility problem associated with Lemma 6 for (9) is quasi-convex in k, so, for a fixed p, we can do a bisection on k to find the largest value that makes the LMI feasible. In [10], $p = 1$ (quadratic), and $p = 2$ (quartic) sum of square Lyapunov functions are considered and largest k for which stability is demonstrated is 5.73. [9] considers the same example, but approaches the problem with Lyapunov functions that are piecewise maximums of quartic sum of squares polynomials, and finds a maximum value of k of 6.2.

The results of taking larger values for p are shown in Figure 1, which gives the maximum value of k for which we can demonstrate stability of the system in (9) as a function of the degree of the sum of squares Lyapunov function. Picking an arbitrary largest value for p of 10, we can demonstrate stability for (9) with $k \leq 6.86$. For this example, using the positive definiteness re-

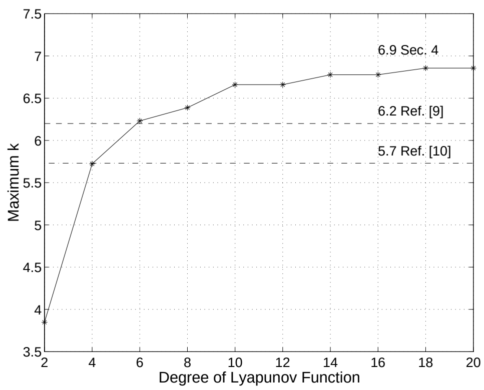

Figure 1: Maximum k for which stability of the time varying system in (9) can be shown for as as a function of the degree of Lyapunov function.

quirement for V from [10] gives, to optimization tolerances, the same results. The ideas in this example easily extend to more plant perturbations, but with more than one perturbation, the bisection technique no longer works.

5.2 Intermittent information observer

Consider the problem of building an observer for system that has two outputs y_0, and y_1, where we are always allowed to use y_0 for the observer, but we only receive y_1 intermittently. An observer for platoon activity can be cast into this form by letting all the onboard sensors be y_0 and letting all the sensor data that comes from all the other vehicles be y_1. y_1 is subject to some non-trivial packet loss since it needs to be transmitted over a possibly very noisy radio link, and can thus be considered intermittent. The question of showing that this observer is stable can be cast into the form of Lemma 8 by looking at the error dynamics of the observer. Writing the plant as

$$\begin{bmatrix} \dot{x} \\ y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C_0 & D_0 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

the observer becomes

$$\begin{aligned} \dot{\hat{x}} = A\hat{x} + Bu - K_0\Big(y_0 - C_0\hat{x} - D_0 u\Big) \\ - a(t)K_1\Big(y_1 - C_1\hat{x} - D_1 u\Big) \end{aligned}$$

where $a(t) \in \{0, 1\}$ tells us if y_1 is being received or not. Defining the observer error as $e = x - \hat{x}$, we have

$$\begin{aligned} \dot{e} &= Ae - K_0C_0e - a(t)K_1C_1e \\ &= \Big[a(t)\Big(A - K_0C_0 - K_1C_1\Big) + (1 - a(t))\Big(A - K_0C_0\Big)\Big]\, e \end{aligned}$$

To establish the stability of this observer's error dynamics we need only follow Lemma 6 for the matrices above, as long as $a(t)$ has only finite discontinuities on finite intervals, which hopefully any transmission network would. It is important to note that in the framework of Lemma 6, we will be solving this problem for all $a(t)$ which will add an additional layer of conservatism to our stability results.

6 Conclusions

The power transformation that forms the heart of this paper has been around for almost thirty years, but due to its complexity and the non-uniqueness it instills in the matrix forms of polynomials, it has seen only sporadic use. The transform proves to be very useful in that it makes considering non-quadratic polynomials as Lyapunov functions easier, and in this form we get the result that one only needs to search over homogeneous polynomials when considering the linear simultaneous stability problem.

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This work is supported by DARPA under the Software Enabled Control program, which is managed by Dr. John Bay. USAF contract # F33615-99-C-1497.