Using the Worstcase Solver - Demo 1

The `worstcase` solver is used to find the induced L2-to-L2 gain of a four-state nonlinear system.

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Contents

Introduction.

Consider a dynamic system of the form

$$\dot{x} = f(x, u)$$

$$y = g(x, u),$$

where `x(0)=0`. Given positive scalars `B` and `T`, the goal is to maximize

$$\|y\|_{2,T} := \int_0^T \|y(t)\|_2 \, dt$$

subject to the constraint

$$\|u\|_{2,T} := \int_0^T \|u(t)\|_2 \, dt \le B.$$

Note: we only consider inputs and outputs defined on the interval `[0,T]`.

System parameters.

This system is parameterized by the following constants:

```
lam = 1;
```

```
PL = 1;
gammaX = 1;
gammaR = 1;
A = 0.8;
tau = 1;
K0x = (-1/tau - A)/lam;
K0r = (1/tau)/lam;
```

Create a model of the system.

First, `polynomial` variables are created using the `pvar` command. Then, these variables are used to define the functions `f` and `g`, which are also `polynomial` variables.

```
pvar x1 xm zx zr r w
states = [x1;xm;zx;zr];
inputs = [r;w];

f(1,1) = A*x1 + lam*((zx+K0x)*x1 + (zr+K0r)*r) + w;
f(2,1) = (1/tau)*(-xm+r);
f(3,1) = -gammaX*x1*(x1-xm)*PL;
f(4,1) = -gammaR*r*(x1-xm)*PL;

g = ((zx+K0x)*x1 + (zr+K0r)*r) + w;
```

Then, a `polysys` object is created from the polynomials `f` and `g`.

```
sys = polysys(f,g,states,inputs);
```

The polynomial objects `states` and `inputs` specify the ordering of the variables. That is, by setting `states(1) = x1`, we specify that `f(1)` is the time derivative of `x1`.

Optimization parameters.

Use the following values for the optimization parameters (defined above):

```
T = 10;
B = 3;
```

The time vector t specifies the time window (T = t(end)) and the points at which the system trajectory is computed.

```
t = linspace(0,T,100)';
```

Set options for worstcase solver.

Create a `wcoptions` object that contains the default options.

```
opt = wcoptions();
```

Specify the maximum number of iterations and which ODE solver to use.

```
opt.MaxIter = 50;
opt.ODESolver = 'ode45';
```

Tell the solver to display a text summary of each iteration.

```
opt.PlotProgress = 'none';
```

Specify the optimization objective, and the bound on the input.

```
opt.Objective = 'L2';
opt.InputL2Norm = B;
```

Find worstcase input.

```
[tOut,x,y,u,eNorm] = worstcase(sys,t,opt);
```

Simulate with worstcase input.

We can only compute the worstcase input over a finite interval of time [0,T]. However, any response of the system that occurs after the input is "shut off" (i.e., u(t) = 0 for t > T) should contribute to our objective. Hence, we compute a more accurate value of the objective by continuing the simulation from the end of the previous trajectory with no input:

```
[te,xe,ye] = sim(sys,tOut,x(end,:)');
td = [tOut;tOut(2:end)+max(tOut)];
yd = [y;ye(2:end)];
```

The objective value over [0,T] is

```
eNorm
```

```
eNorm =
    4.7436
```

The objective value over [0,2T] is

```
eNormd = get2norm(yd,td)
```

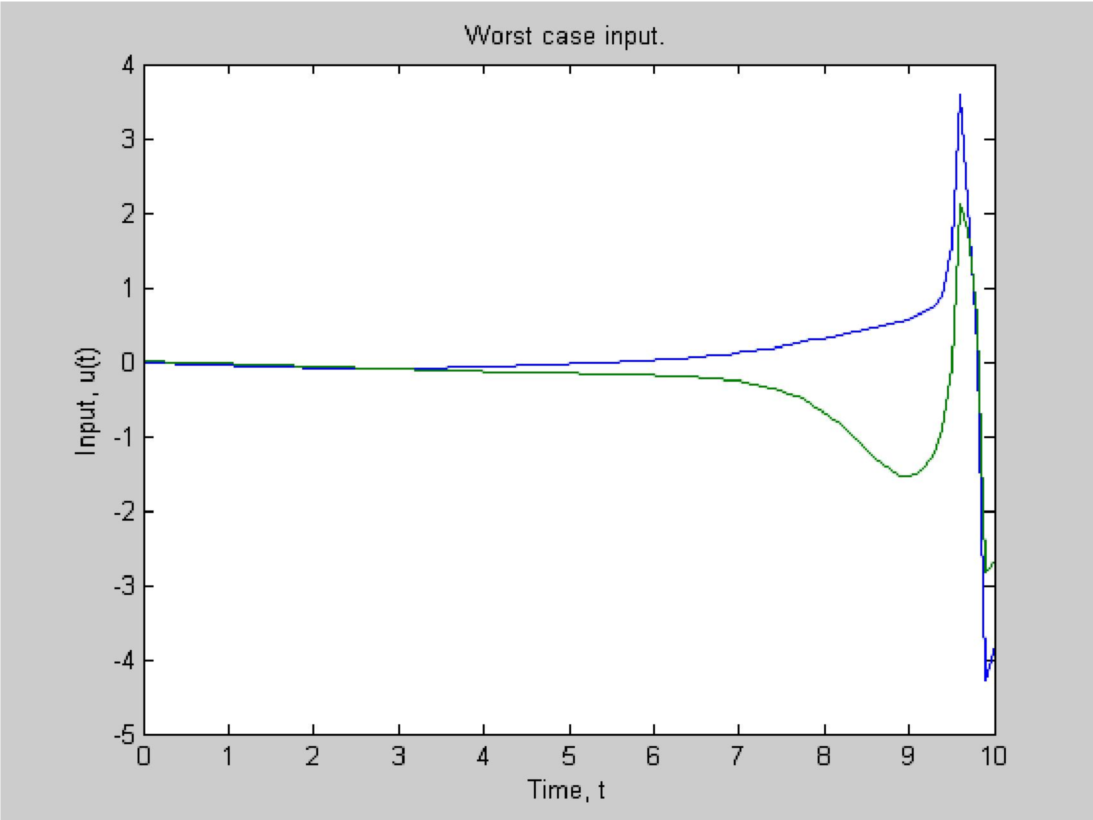
```
eNormd =
    4.9622
```

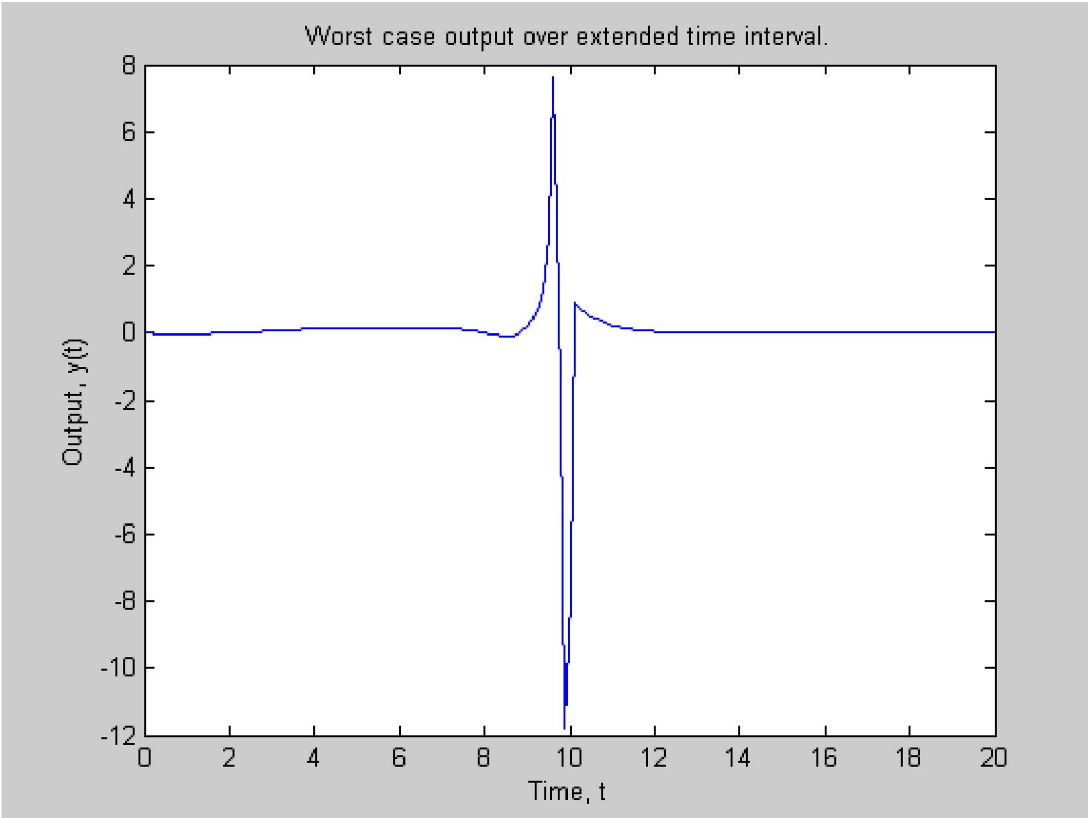
Display results.

```
fprintf( 'The L2-to-L2 gain is %f\n', eNormd/B );

figure;
plot(tOut,u)
```

```
xlabel('Time, t')
ylabel('Input, u(t)')
title('Worst case input.')

figure;
plot(td,yd)
xlabel('Time, t')
ylabel('Output, y(t)')
title('Worst case output over extended time interval.')
```

```
The L2-to-L2 gain is 1.654050
```





Specifying a starting point.

By default, the worstcase solver starts with a constant input and then searches for a better input. Since this problem is nonconvex, this search may get "stuck" at a local optimum. We can help the solver by specifying a sensible starting point that is known to exhibit a large output.

```
load demo1_badInput
u0 = B * ubad/get2norm(ubad,tbad);
opt.InitialInput = u0;
```

Run solver again.

```
[tOut,x,y,u,eNorm] = worstcase(sys,t,opt);
```

Extend this simulation.

```
[te,xe,ye] = sim(sys,tOut,x(end,:)');
td = [tOut;tOut(2:end)+max(tOut)];
yd = [y;ye(2:end)];
```

The objective value over [0,T] is

```
eNorm
```

```
eNorm =
```

```
    5.0020
```

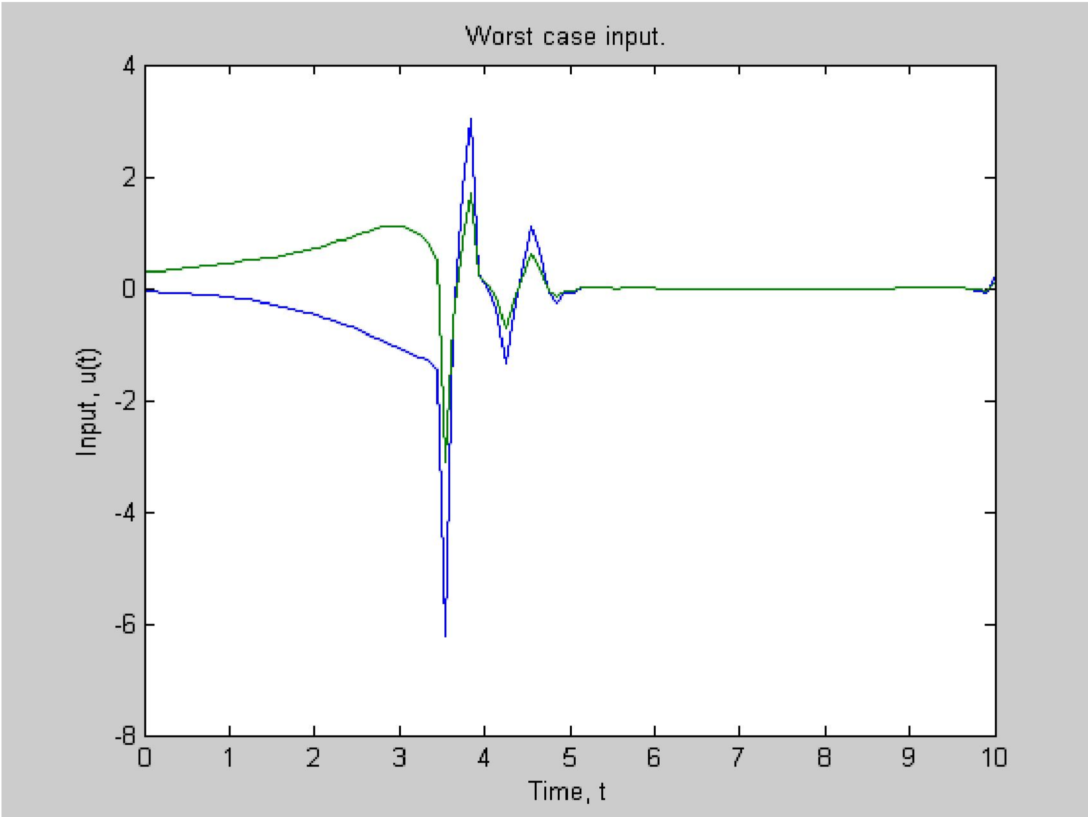
The objective value over [0,2T] is

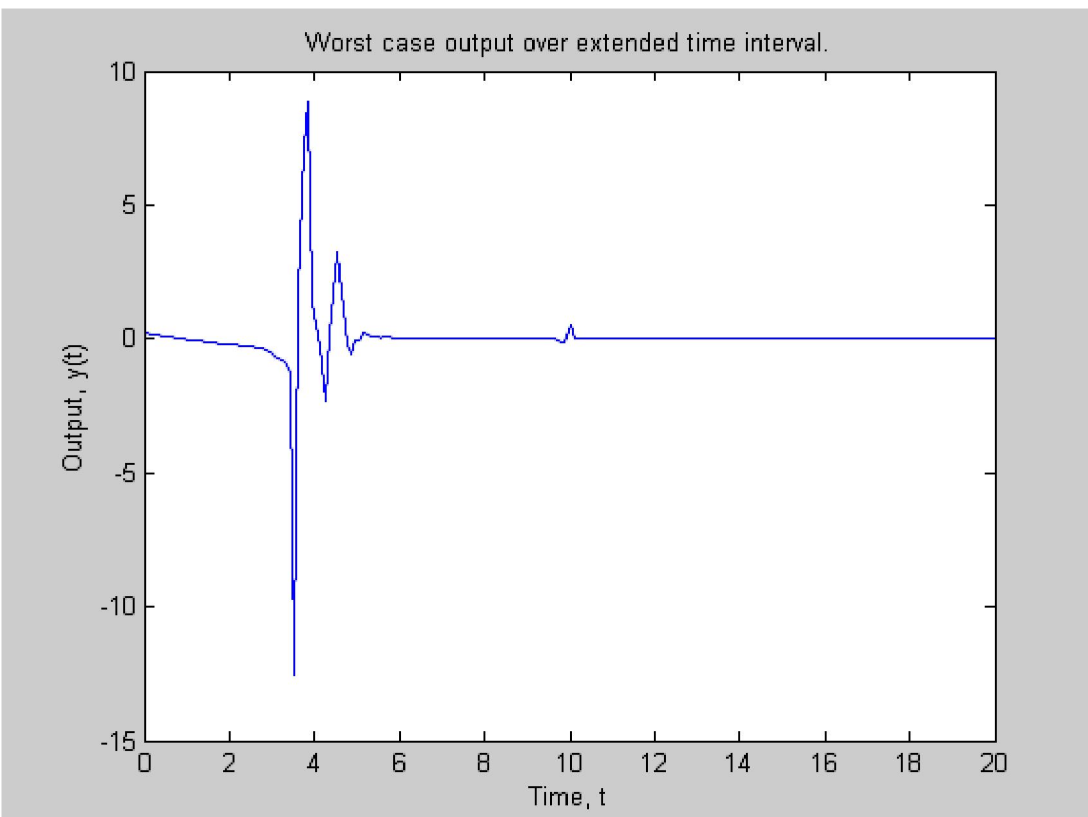
```
eNormd = get2norm(yd,td)


eNormd =
    5.0029
```

Note that we achieve a larger value of the objective when we start the solver at u0.

Display new results.

```
fprintf( 'The L2-to-L2 gain is %f\n', eNormd/B );

figure;
plot(tOut,u)
xlabel('Time, t')
ylabel('Input, u(t)')
title('Worst case input.')

figure;
plot(td,yd)
xlabel('Time, t')
ylabel('Output, y(t)')
title('Worst case output over extended time interval.')


The L2-to-L2 gain is 1.667635
```


Worst case input.
Input, u(t)
Time, t
4
2
0
-2
-4
-6
-8
0
1
2
3
4
5
6
7
8
9
10


Worst case output over extended time interval.
Output, y(t)
Time, t
10
5
0
-5
-10
-15
0
2
4
6
8
10
12
14
16
18
20